2018 Kansas MAA Undergraduate Mathematics Competition

Instructions:

1. This is a *team competition*. You are permitted to work with the members of your team on the following 10 problems. You are not permitted to ask for or receive any assistance from anyone other than your team mates.
2. Each team should submit at most one solution to each problem. Each solution should be completed on a separate sheet of paper. Also, each solution page should the have problem number **AND** the teams number written on it (your team number is located in the upper right hand corner of this paper). Please do not write your school name, the names of the team mates, or any other identifying information on the solutions.
3. You have until 11am to complete the exam. When you are finished with the exam, each team should place the problem sheet and each solution page (with the team number and problem number written on it) into the exam envelope. Any scrap or extra paper should be returned to the graders.
4. Calculators are permitted. However, you must show all of your work for full credit.
5. No cell phones or other electronic devices (other than calculators) are permitted during the exam.

Problems:

1. Let $f(n)$ denote the n-th term in the sequence
$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$$
obtained by writing one 1, two 2's, three 3's, four 4's, etc. Find, with proof, $f(2018)$.
2. Given two numbers $b, c \in [0, 2]$ chosen randomly, what is the probability that the polynomial equation
$$x^2 + bx + c = 0$$
has at least one real root?
3. Find the point in the first quadrant on the graph of $y = 9 - x^2$ such that the distance between the x- and y- intercepts of the tangent line at the point is minimum.
4. Prove that the product of any 2018 consecutive integers is divisible by $2018!$.

5. Find the value of the sum

$$\sum_{n=2}^{\infty} \frac{n^2 - 2n - 4}{n^4 + 4n^2 + 16}.$$

Hint: Notice that $n^4 + 4n^2 + 16 = (n^2 - 2n + 4)(n^2 + 2n + 4)$.

6. Define a family of curves by

$$S_n = \{(x, y) : y = \frac{1}{n} \sin(n^2 x),\ 0 \leq x \leq \pi\},$$

where n is a positive integer. What is the limit of the length of S_n as $n \to \infty$?

7. Let $n \geq 3$ points be given in the plane, so that no three are collinear. Prove that three of them form an angle which is at most π/n.

8. Let $f(x)$ and $g(x)$ be nonzero polynomials with real coefficients such that

$$f(x^2 + x + 1) = f(x)g(x)$$

for all $x \in \mathbb{R}$. Show that f must have even degree.

9. Determine, with proof, for which $n \in \mathbb{N}$ is $7^n + 147$ is a perfect square.

10. Let $x = 0.0102040816326428\ldots$ be the real number in the interval $(0, 1)$ whose decomal expansion (after the decimal point) is obtained by concatenating the last two digits of the sequence of powers of 2 (padded by 0 if there is only one digit). Is x a rational or irrational number? Prove your claim!