Problems for the First KS math competition

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- **Problem 1**
  Let f be a continuous function on $[0,1]$, such that for every $x \in [0,1]$, $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$. Show that
  $$\int_0^1 f^2(x)dx \geq \frac{1}{3}.$$

  Solution:
  $$0 \leq \int_0^1 (f(x)-x)^2 dx = \int_0^1 f^2(x)dx - 2\int_0^1 xf(x)dx + \int_0^1 x^2 dx.$$
  It follows
  $$\int_0^1 f^2(x)dx \geq 2\int_0^1 xf(x)dx - \frac{1}{3}.$$
  But
  $$\frac{1}{3} = \int_0^1 \frac{1-x^2}{2} dx \leq \int_0^1 (\int_0^t dx) f(t) dx dt = \int_0^1 tf(t)dt,$$
  whence
  $$\int_0^1 f^2(x)dx \geq \frac{1}{3}.$$

- **Problem 2** Let $x_{n+1} = \frac{4}{2-x_n}$, where $x_0 = 1$. Determine

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k.$$

Solution The sequence is periodic with period 3: $x_0 = 1$, $x_1 = 4$, $x_2 = -2$ and $x_3 = 1$. It follows that $S_n = \sum_{k=1}^{n} x_k$ is

$$S_n = \begin{cases} 3m & n = 3m \\ 3m+4 & n = 3m+1 \\ 3m+2 & n = 3m+2 \end{cases}$$

It is clear that $1 \leq S_n/n \leq (n+3)/n$ and the $\lim_n S_n/n = 1$.

- **Problem 3** Let P be a polynomial of degree n with real coefficients and real zeros only. Show that

$$(n-1)(P'(x))^2 \geq nP(x)P''(x).$$

When do you achieve equality for all x? **Solution:** Since $P(x) = a(x-x_1)\ldots(x-x_n)$, we have

$$\frac{P'(x)}{P(x)} = \sum_{j=1}^{n} \frac{1}{x-x_j}$$
$$\frac{P''(x)}{P(x)} = \sum_{1\leq i<j\leq n} \frac{2}{(x-x_j)(x-x_i)}$$

Thus

$$(n-1)\left(\frac{P'(x)}{P(x)}\right)^2 - n\frac{P''(x)}{P(x)} = \sum_{j=1}^{n} \frac{(n-1)}{(x-x_j)^2} - \sum_{1\leq i<j\leq n} \frac{2}{(x-x_j)(x-x_i)} =$$
$$= \sum_{1\leq i<j\leq n} \left(\frac{1}{x-x_i} - \frac{1}{x-x_j}\right)^2 \geq 0.$$

- **Problem 4**

  Find all differentiable functions $F : R^+ \to R^+$, so that

  $$f(x)f(yf(x)) = f(x+y).$$

  Solution:

  Write the condition as

  $$f^2(x)\frac{f(yf(x))-1}{yf(x)} = \frac{f(x+y)-f(x)}{y}$$

  Take a limit as $y \to 0$ to get $f'(x) = -f'(0)f^2(x)$, which gives the solution $f(x) = \frac{1}{(ax+b)}$. Plug this in the original equation to find that only when $b = 1$, this will be satisfied.

- **Problem 5**

  Let A and B are two $n \times n$ symmetric matrices with real entries, which do not necessarily commute. Assume also that A is positive in the sense that all eigenvalues are positive. Show that AB has all eigenvalues real.

  Solution: Since A is symmetric and positive, then $A = T^{-1}KT$, where K is diagonal with positive entries $\lambda_1, \ldots, \lambda_n$ on the diagonal and T is invertible matrix. Define $K_{1/2}$, to be the diagonal matrix with $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$ on the diagonal and $C = T^{-1}K_{1/2}T$ is invertible. Clearly $K_{1/2}^2 = K$ and $C^2 = A$. We have

  $$C^{-1}ABC = C^{-1}C^2BC = CBC.$$

  It is clear that CBC is symmteric with real entries $(CBC)^t = C^tB^tC^t = CBC$ and therefore has only real eigenvalues. But AB is similar to CBC and therefore has the same *real* eigenvalues.

- **Problem 6**

Let A be a real 4×2 matrix, while B is real 2×4 matrix. We know

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Find BA.

Solution:

Represent $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $B = (B_1, B_2)$, where A_1, A_2, B_1, B_2 are 2×2 matrices. We have

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} (B_1, B_2) = \begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{pmatrix}.$$

It follows that $A_1B_1 = A_2B_2 = I$ and $A_1B_2 = A_2B_1 = -I$. Then

$$BA = (B_1, B_2) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = B_1A_1 + B_2A_2 = 2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

- **Problem 7**

Let $p_1, \ldots, p_n$ be finitely many points in the unit ball. Show that there exists at least one point on the unit circle p, so that

$$\frac{1}{n} \sum_{k=1}^{n} |p - p_i| \geq 1.$$

Solution: Choose p to be the unit vector in the direction opposite to $p_1 + \ldots + p_n$. We have by the triangle inequality

$$\sum_{j=1}^{n} |p - p_j| \geq |np - \sum_{j=1}^{n} p_j| = n + |\sum_{j=1}^{n} p_j| \geq n.$$

- **Problem 8**
  Let $z \neq 0$ and A and B are two matrices, with

$$AB - BA = zA$$

Show that for all integers k, $A^k B - BA^k = zkA^k$. Show that there exists k, so that $A^k = 0$.
Solution: We have

$$\begin{aligned} A^k B - BA^k &= \sum_{j=1}^{k} (A^{k-j+1} B A^{j-1} - A^{k-j} B A^j) = \\ &= \sum_{j=1}^{k} A^{k-j}(AB - BA)A^{j-1} = \sum_{j=1}^{k} A^{k-j} z A A^{j-1} = zkA^k. \end{aligned}$$

For the second part, it is equivalent to show that A has only zero eigenvalues. Suppose not. Assume without loss of generality (by rescaling) that A has eigenvalues, satisfying $|\lambda| \leq 1$ and an eigenvalue $\lambda_0 : |\lambda_0| = 1$. It is clear now that the entries of A^k are uniformly bounded in k, whence the entries of $A^k B - BA^k$ are uniformly bounded in k. The right-hand zkA^k has entries that increase linearly with k and that is a contradiction.