Second KS math competition

March 21, 2006

- **Problem 1: 10 points**

  Compute the limit
  $$\lim_{N\to\infty}\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right).$$

  Solution Compute
  $$S_N=\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right)=\prod_{n=2}^{N}\frac{(n-1)(n+1)}{n^2}=\frac{N+1}{2N}.$$

  So, $\lim_{N\to\infty} S_N = 1/2$.

- **Problem 2: 10 points**

  Compute
  $$\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}\,dt$$

  Solution: Rewrite
  $$x\int_0^x e^{t^2-x^2}\,dt=\frac{\int_0^x e^{t^2}\,dt}{e^{x^2}/x}.$$

  It is clear that
  $$\lim_{x\to\infty}\int_0^x e^{t^2}\,dt=\infty$$
  $$\lim_{x\to\infty} e^{x^2}/x=\infty.$$

  By the L'Hospital's rule,
  $$\lim_{x\to\infty}\frac{\int_0^x e^{t^2}\,dt}{e^{x^2}/x}=\lim_{x\to\infty}\frac{e^{x^2}}{\frac{2x^2e^{x^2}-e^{x^2}}{x^2}}=\lim_{x\to\infty}\frac{x^2e^{x^2}}{2x^2e^{x^2}-e^{x^2}}=1/2.$$

- **Problem 3: 10 points**

  A fair coin is tossed repeatedly. What is the probability that the first head will appear in an odd try? In other words, the positive outcomes are: the first try is a head or the first two are tails and then the third one is a head or the first four are tails and then the fifth one is a head etc.
  Solution: The favorable events can be decomposed in a sum of disjoint events as follows: head in the first try or two tails first and then head or four tails first and then head and so on. The corresponding probabilities are

  $$p = 1/2 + (1/2)^3 + (1/2)^5 + \ldots = \frac{1}{2}\frac{1}{(1-1/4)} = 2/3.$$

- **Problem 4: 10 points**
  Compute

  $$\lim_{n\to\infty} n\left[\frac{1}{n^2+1} + \frac{1}{n^2+2^2} + \ldots + \frac{1}{n^2+n^2}\right]$$

  Solution:
  Write

  $$n\left[\frac{1}{n^2+1} + \frac{1}{n^2+2^2} + \ldots + \frac{1}{n^2+n^2}\right] = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+(k/n)^2}.$$

  The last expression is a Riemann sum for the integral

  $$\int_0^1 \frac{1}{1+x^2}dx.$$

  Consequently, the limit is

  $$\int_0^1 \frac{1}{1+x^2}dx = \tan^{-1}(x)|_0^1 = \pi/4.$$

- **Problem 5: 10 points**
  Five points in the plane belong to a closed square with side 1. Prove that the distance between some two of them is at most $\sqrt{2}/2$.
  Solution:
  Divide the unit square in four identical subsquares with sidelength 1/2 passing through the center. By the pigeonhole principle, there will be two points in one of the squares (including its sides). Therefore, the distance between them will not be more than the diameter of each of the subsquares which is $\sqrt{2}/2$.