Third KS math competition

April 3, 2007

1. Show that for every sequence $x_1, \ldots, x_n \in (0,1)$ *at least one of the inequalities holds:*

   $$x_1 \ldots x_n \leq 2^{-n}$$

   or

   $$(1-x_1)\ldots(1-x_n) \leq 2^{-n}$$

   Solution: Suppose the claim is not true. Then

   $$x_1 \ldots x_n(1-x_1)\ldots(1-x_n) > 4^{-n}$$

   But $x_1(1-x_1) \leq 1/4, \ldots, x_n(1-x_n) \leq 1/4$, we get

   $$4^{-n} < x_1 \ldots x_n(1-x_1)\ldots(1-x_n) \leq 4^{-n},$$

   a contradiction.

2. Compute

   $$L = \lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2+i^2)^{1/n}$$

   Solution: Let $A_n = n^{-4}\prod_{i=1}^{2n}(n^2+i^2)^{1/n}$. We have

   $$\begin{aligned}\ln(A_n) &= \left(\sum_{i=1}^{2n} \frac{1}{n}\ln(n^2+i^2)\right) - 4\ln(n) = \left(\sum_{i=1}^{2n} \frac{1}{n}\ln(n^2(1+i^2/n^2)\right) - 4\ln(n) = \\ &= \left(\sum_{i=1}^{2n} \frac{1}{n}(2\ln(n)+\ln(1+i^2/n^2))\right) - 4\ln(n) = \sum_{i=1}^{2n} \frac{1}{n}\ln(1+i^2/n^2)\end{aligned}$$

The last expression is a Riemann sum for $\int_0^2 \ln(1+x^2)dx$ and therefore $\lim_n A_n = exp(\int_0^2 \ln(1+x^2)dx)$ and

$$\int_0^2 \ln(1+x^2)dx = x\ln(1+x^2)|_0^2 - 2\int_0^2 \frac{x^2}{(1+x^2)}dx =$$
$$= 2\ln(5) - 2 + 2\tan^{-1}(2).$$

Thus $L = e^{2\ln(5)-2+2\tan^{-1}(2)} = 25e^{2\tan^{-1}(2)-2}$.

3. Find the limit of the sequence

$$\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \dots$$

Solution: Denote the n^{th} term of the sequence by x_n. We have $x_{n+1} = \sqrt{1+x_n}$, $x_1 = 1$. We establish by induction that $x_{n+1} > x_n$. Indeed, $x_2 = \sqrt{2} > x_1 = 1$. Assuming $x_n > x_{n-1}$, we get

$$x_{n+1} = \sqrt{1+x_n} > \sqrt{1+x_{n-1}} = x_n.$$

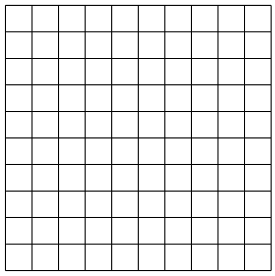
Next, we establish by induction that $x_n < 2$ (Any number bigger than $(1+\sqrt{5})/2$ will do here). Indeed, $x_1 < 2$ and assuming $x_n < 2$, we get $x_{n+1} = \sqrt{1+x_n} < \sqrt{3} < 2$.

Thus, x_n is increasing and bounded, thus convergent. Denote $L = \lim_n x_n$. Then $L = \sqrt{1+L}$, whence $L = (1+\sqrt{5})/2$.

4. A coin is tossed 10 times. Find the probability of not having two consecutive tails.

Solution: Solve more genral problem with n tosses instead of 10 tosses. Record the outcomes with 0 if tail turns up and 1 otherwise. Thus, we are counting the numbers of sequences of $0, 1$, which will not have two consecutive 0. Denote the number of these (let us call them favorable sequences) f_n. Any sequence like that will either start with zero or one. If it starts with 1, we may concatenate this with any favorable sequence of length $n-1$. If it starts with a zero, then we must have 1 in the next slot, after which, we may concatenate with any favorable sequence with length $n-2$. Thus $f_n = f_{n-1} + f_{n-2}$. Also, see that $f_1 = 2, f_2 = 3$. Thus, $f_{10} = 144$. The total number of outcomes is $2^{10} = 1024$, and thus the probability is $144/1024$.

5. How many squares (of all possible sizes) are there in the following picture?



Solution: Count the number of possible lower left end corners of squares with sidelength $k, 1 \leq k \leq 10$. We have clearly the lower left square with sidelength $(10-k)$, this yields $(10-k+1)^2$ points. Thus the number of squares is

$$\sum_{k=1}^{10}(10-k+1)^2 = 385.$$