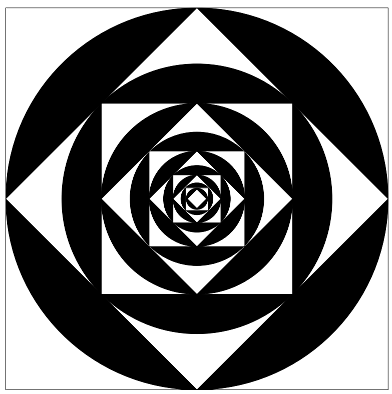
Problem 1. If $\lim_{x\to\infty}\left(\frac{x+2a}{x+a}\right)^x = 8$, find a.

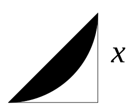
Solution. $\lim_{x\to\infty}\left(\frac{x+2a}{x+a}\right)^x = \lim_{x\to\infty}\left(\frac{1+2a/x}{1+a/x}\right)^x = \frac{\lim_{x\to\infty}\left(1+2\frac{a}{x}\right)^x}{\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x} = \frac{e^{2a}}{e^a} = e^a = 8$

so $\boxed{\boldsymbol{a = \ln 8.}}$

Problem 2. The following figure consists of infinitely many squares and circles, with a circle inscribed in each square and a square inscribed in each circle. The outermost square has side length 1. Find the total shaded area.



Solution. The figure can be decomposed into infinitely many figures similar to the following:



The total area is $x^2/2$. The shaded area A is one-fourth of the difference of a circle of radius x and a square of side length $x\sqrt{2}$, i.e., $A = (\pi x^2 - 2x^2)/4$. Therefore the, shaded area accounts for

$$\frac{(\pi x^2 - 2x^2)/4}{x^2/2} = \frac{\pi - 2}{2}$$

of the total area of the original figure. The total area is 1, so the shaded area is $\boxed{\boldsymbol{(\pi - 2)/2.}}$

Problem 3. Find the determinant of the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} (-1)^{|i-j|} & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

Solution. The matrix initially looks like this:

$$\begin{bmatrix} 2 & -1 & 1 & \cdots & (-1)^n & (-1)^{n-1} \\ -1 & 2 & -1 & \cdots & (-1)^{n-1} & (-1)^n \\ 1 & -1 & 2 & \cdots & (-1)^n & (-1)^{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-1)^n & (-1)^{n-1} & (-1)^n & \cdots & 2 & -1 \\ (-1)^{n-1} & (-1)^n & (-1)^{n-1} & \cdots & -1 & 2 \end{bmatrix}$$

For each even $i > 1$, add the first row to the i^{th} row, and for each odd $i > 1$, subtract the first row from the i^{th} row. The result is the matrix

$$\begin{bmatrix} 2 & -1 & 1 & \cdots & (-1)^n & (-1)^{n-1} \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-1)^{n-1} & 0 & 0 & \cdots & 1 & 0 \\ (-1)^n & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Now, for each even $i > 1$, add the i^{th} row to the first row, and for each odd $i > 1$, subtract the i^{th} row from the first row. The result is the matrix

$$\begin{bmatrix} n+1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-1)^{n-1} & 0 & 0 & \cdots & 1 & 0 \\ (-1)^n & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

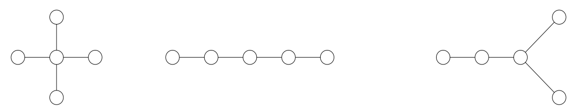
whose determinant is evidently $\boxed{\boldsymbol{n+1.}}$

Problem 4. A Martian colony consists of five islands (Alpha, Beta, Gamma, Delta and Epsilon) floating in space. Four bridges are to be built to enable travel between any pair of islands.

(a) In how many ways is this possible?

(b) Suppose that there is already a bridge between Beta and Gamma. How many possibilities are there for the remaining three bridges?

Solution. (a) The bridges form a tree with one of the following three topologies.



— There are 5 ways to label the tree shown on the left (choose the center, and then the others are determined).
— There are 60 ways to label the tree in the middle (choose a permutation of $\{A, B, C, D, E\}$, up to reversal).
— There are 60 ways to label the tree on the right (choose the three leftmost vertices in $5 \times 4 \times 3$ ways).

Therefore, the answer is $5+60+60 = \boxed{\mathbf{125.}}$

(b) There are 10 possible bridges, and each tree contains 4 bridges. By symmetry, each bridge appears in the same number of trees, namely $(4/10)(125) = \boxed{\mathbf{50.}}$

(Of course, it is also possible to count the trees directly, but this is much easier.)

Problem 5. Let $f(x)$ be a function that is continuously differentiable on $[0,1]$, such that $f(0) = 0$, $f(1) = 1$, and $x < f(x) < 1$ for all x, $0 < x < 1$.

(a) Prove that there exist two distinct points c_1, c_2 in $(0,1)$ such that $f'(c_1)f'(c_2) = 1$.

(b) For extra credit: Prove that for every positive integer n, there exist n distinct points $c_1, c_2, \ldots, c_n$ in $(0,1)$ such that $f'(c_1)f'(c_2)\cdots f'(c_n) = 1$.

Solution 1. Let f_n denote the composition of f with itself n times, i.e.,

$$f_1 = f, \quad f_2 = f \circ f, \quad f_3 = f \circ f \circ f,$$

et cetera. Each function f_n is continuously differentiable on $[0,1]$, with $f_n(0) = 0$, $f_n(1) = 1$. By the Mean Value Theorem, there exists $c \in (0,1)$ such that

$$f_n'(c) = \frac{f_n(1) - f_n(0)}{1 - 0} = 1.$$

By the Chain Rule,

$$f_n'(c) \;=\; 1 \;=\; f'(f_{n-1}(c)) \cdot f'(f_{n-2}(c)) \cdot \cdots \cdot f'(f(c)) \cdot f'(c).$$

So we can define $c_1, \ldots, c_n$ by $c_1 = c$, $c_2 = f(c)$, $c_3 = f(f(c))$, $\ldots$, $c_n = f_{n-1}(c)$.

The condition that $x < f(x) < 1$ for $x \in (0,1)$ implies that $0 < c_1 < c_2 < \cdots < c_n < 1$. In particular the c_i are distinct.

Solution 2. By the Mean Value Theorem, there exists $x \in [0,1]$ such that $g'(x) = (g(1) - g(0))/(1 - 0) = 1$. By continuity, there exists some $\epsilon > 0$ such that g' takes on the values $1 + \epsilon$ and $1/(1+\epsilon)$ in some neighborhood of x within $(0,1)$. More generally, there exists some $\epsilon > 0$ such that g' takes on the values

$$1 + \varepsilon, \quad 1 + \varepsilon^2, \quad \ldots, \quad 1 + \varepsilon^{n-1}, \quad \prod_{j=1}^{n-1} (1 + \varepsilon^j)^{-1}$$

in some neighborhood of x, necessarily on distinct points.

(Other solutions are possible.)