Fifth Annual Kansas Mathematics Competition
May 2, 2009

Problem 1. Consider a sequence of integers $1, 3, 2, -1, \dots$, where each term is equal to the term preceding it minus the term before that. What's the sum of the first 2009 terms?

Solution. The sequence repeats: $1, 3, 2, -1, -3, -2, 1, 3, 2, \dots$ Each six conseuctive terms sum up to 0. Since 2009 is congruent to 5 mod 6, the first 2009 terms have the same sum as the first 5 terms: $1 + 3 + 2 - 1 - 3 = 2$. So the answer is 2.

Problem 2. How many ways are there to assign the labels A, B, C, D, E, F to the vertices of a hexagon so that none of the pairs AB, CD, or EF form an edge?

Solution. The chance that AB forms an edge is $2/5$, so the number of such assignments is $2/5 \cdot 6! = 288$. The same is true if we replace AB wiith CD or with EF.

The chance that two of the forbidden pairs form edges is $2/5 \cdot 1/2 = 1/5$, so the number of such assignments is $1/5 \cdot 6! = 144$.

The chance that all three form edges is $2/5 \cdot 1/2 \cdot 2/3 = 2/15$, so the number of such assignments is $2/15 \cdot 6! = 96$.

By inclusion/exclusion, the answer is therefore

$$720 - 3 \cdot 288 + 3 \cdot 144 - 96 = 192.$$

Alternate solution. Break the possible assignments into cases as follows.

<u>Case 1:</u> A, B are opposite each other. This can happen in 6 ways. Of the 4!=24 ways of assigning the other four labels to vertices, one-third (i.e., 8) of them have CD and EF as edges; none of the others do. So this case accounts for $6 \cdot (24 - 8) = 6 \cdot 16 = 96$ labelings.

<u>Case 2:</u> A, B have a common neighbor. This can happen in 12 ways (since there are 6 possible locations for A, and then 2 possibilities for B). There are 4 possibilities for the common neighbor x, and then x must be opposite the label which it is forbidden to neighbor (for example, if D is between A and B, then C is opposite D), leaving 2 possibilities for the remaining two vertices. This accounts for $12 \cdot 4 \cdot 2 = 96$ labelings.

So the grand total is $96 + 96 + 192$.

(There are other ways of solving the problem by breaking it into cases.)

Problem 3. Two players, A and B, play a game with a fair six-sided die. The goal is to roll a 2 or a 5: whoever does so first wins the game. The players take turns rolling the die, with player A going first. They keep rolling until someone rolls a 2 or a 5. What is the probability that player A wins the game?

Solution. The probability that the game lasts at least n turns is $(2/3)^{n-1}$. Therefore, the probability that the game lasts *exactly* n turns is $(2/3)^{n-1} - (2/3)^n$. To get the probability that A wins, sum this expression for all positive odd n, i.e.,

$$1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \cdots$$

This is a geometric series with initial term 1 and ratio of successive terms equal to $-2/3$, so its sum is

$$\frac{1}{1+\frac{2}{3}} = \frac{3}{5}.$$

Problem 4. Let $0 < a < b$. Evaluate

$$\lim_{p\to 0} \left(\int_0^1 (bx + a(1-x))^p \, dx \right)^{\frac{1}{p}}.$$

Solution. To evaluate the integral, make the change of variable $u = bx + a(1-x)$, $du = (b-a)dx$ to get

$$\left(\frac{1}{b-a} \int_a^b u^p \, du \right)^{\frac{1}{p}} = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}} = \exp\left(\frac{1}{p} \ln \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right) \right).$$

Now, to evaluate the limit as $p \to 0$, use L'Hôpital's rule:

$$\begin{aligned}
&\exp \lim_{p\to 0} \frac{\ln\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)}{p} \\
&= \exp \lim_{p\to 0} \frac{(p+1)(b-a)}{b^{p+1}-a^{p+1}} \frac{(p+1)(b-a)(b^{p+1}\ln b - a^{p+1}\ln a) - (b^{p+1}-a^{p+1})(b-a)}{(p+1)^2(b-a)^2} \\
&= \exp \lim_{p\to 0} \frac{(p+1)(b^{p+1}\ln b - a^{p+1}\ln a) - (b^{p+1}-a^{p+1})}{(b^{p+1}-a^{p+1})(p+1)} \\
&= \exp \lim_{p\to 0} \left(\frac{b^{p+1}\ln b - a^{p+1}\ln a}{b^{p+1}-a^{p+1}} - 1 \right) \\
&= \exp \left(\frac{b\ln b - a\ln a}{b-a} - 1 \right) \\
&= e^{-1} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}.
\end{aligned}$$

Problem 5. Let r be a real number. Prove that r is rational if and only if there exist three distinct integers a, b, c such that

$$\frac{r+a}{r+b} = \frac{r+b}{r+c}.$$

Solution. First, note that

$$\begin{aligned} \frac{r+a}{r+b} = \frac{r+b}{r+c} &\iff (r+a)(r+c) = (r+b)^2 \\ &\iff r^2 + (a+c)r + ac = r^2 + 2br + b^2 \\ &\iff (a+c-2b)r = b^2 - ac \\ &\iff r = \frac{b^2 - ac}{a+c-2b}. \end{aligned}$$

This takes care of one direction — if such a, b, c exist then r is rational.

On the other hand, suppose r is rational, say $r = p/q$ with p, q integers, $q > 0$. If $p = 0$ then there are lots of solutions for a, b, c. If $p \neq 0$, then

$$\frac{p}{q}, \qquad \frac{p}{q}(1+q) = r + p, \qquad \frac{p}{q}(1+q)^2 = r + (2p + pq)$$

so we can take $a = 0$, $b = p$, $c = 2p + pq$.