Sixth KS math competition

March 27, 2010

1. Show that for every $0 < \theta \leq \pi$, one has
$$\int_0^\theta \sqrt{1+\cos^2(t)}dt > \sqrt{\theta^2+\sin^2(\theta)}.$$
Solution: The problem is very easy, if one realizes that the left hand side is the arc length of the curve $y = \sin(x)$ from $(0,0)$ to $(\theta, \sin(\theta))$, while the right hand side is the Euclidian distance between the same points.
Analytical Solution Define the function
$$F(\theta) = \int_0^\theta \sqrt{1+\cos^2(t)}dt - \sqrt{\theta^2+\sin^2(\theta)}.$$
Since $F(0) = 0$, it will suffice to show that $F'(\theta) > 0$. We have (by the fundamental theorem of calculus)
$$F'(\theta) = \sqrt{1+\cos^2(\theta)} - \frac{\theta + \sin\theta\cos\theta}{\sqrt{\theta^2+\sin^2(\theta)}}.$$
Thus, matters amount to showing
$$(1+\cos^2(\theta))(\theta^2+\sin^2(\theta)) > (\theta+\sin\theta\cos\theta)^2.$$
The last one is equivalent (after squaring) to
$$\theta^2\cos^2(\theta) + \sin^2(\theta) > 2\theta\sin\theta\cos\theta,$$
which is equivalent to $(\theta\cos(\theta) - \sin\theta)^2 > 0$, which is satisfied for all $\theta \in (0, \pi]$.

2. How many ordered triples of integers (x, y, z) satisfy the equation $|x|+|y|+|z| = 2010$?

Solution #1:

First of all, the number of integer solutions of $|x| + |y| \le n$ is

$$f(n) = 1 + 3 + 5 + \cdots + (2n-1) + (2n+1) + (2n-1) + \cdots + 3 + 1$$

(by counting the number of solutions with $x = -n$, $x = -n+1$, $\ldots$, $x = 0$, $\ldots$, $x = n$). Note that

$$\begin{aligned} f(n) &= (1 + 3 + 5 + \cdots + (2n-1) + (2n+1)) + (1 + 3 + 5 + \cdots + (2n-1)) \\ &= (n+1)^2 + n^2 = 2n^2 + 2n + 1. \end{aligned}$$

Now let $g(n)$ be the number of integer solutions of $|x| + |y| + |z| \le n$. Then counting the number of solutions with $x = -n$, $x = -n+1$, $\ldots$, $x = 0$, $\ldots$, $x = n$ gives

$$\begin{aligned} g(0) &= 1, \\ g(1) &= 1 + (1+3+1) + 1, \\ g(2) &= 1 + (1+3+1) + (1+3+5+3+1) + (1+3+1) + 1, \\ \cdots g(n) &= f(0) + f(1) + \cdots + f(n-1) + f(n) + f(n-1) + \cdots + f(1) + f(0). \end{aligned}$$

In particular,

$$g(n) - g(n-1) = f(n) + f(n-1) = 2n^2 + 2n + 1 + 2(n-1)^2 + 2(n-1) + 1 = 4n^2 + 2$$

and the number we're looking for is

$$g(2010) - g(2009) = 4(2010^2) + 2 = 4(4040100) + 2 = 16160402.$$

Solution #2: Let $h(n, d)$ be the number of *positive* integer solutions of $x_1 + \cdots + x_d = n$. This is the same as the number of non-negative integer solutions of $x_1 + \cdots + x_d = n - d$. This is the number of ways of arranging $n-d$ stars and $d-1$ bars and taking x_i to be the number of stars in the i^{th} block: e.g., $**||****|*|$ corresponds to 2+0+4+1+0=7. Therefore

$$h(n, d) = \binom{n-1}{d-1}.$$

If we count the solutions to $|x|+|y|=|z|=n$ by the number of 0s they contain, we get

$$\begin{aligned} 6h(n,1)+12h(n,2)+8h(n,3) &= 6\binom{n}{0}+12\binom{n-1}{1}+8\binom{n-1}{2} \\ &= 6+12(n-1)+8(n-1)(n-2)/2 \end{aligned}$$

and cleaning up this expression gives $4n^2+2$ as in Solution #1.

3. Let f_n denote the n^{th} Fibonacci number: $f_0=0$, $f_1=1$, and $f_n=f_{n-1}+f_{n-2}$ for $n\geq 2$. Evaluate

$$\sum_{n=1}^{\infty}\frac{f_n}{c^n}.$$

Solution: Call this number A. Set $f_0=0$ for convenience (so in fact $f_2=f_1+f_0$). Then:

$$\begin{aligned} A &= \sum_{n=1}^{\infty}\frac{f_n}{c^n} \\ &= \frac{1}{c}+\sum_{n=2}^{\infty}\frac{f_{n-1}+f_{n-2}}{c^n} \\ &= \frac{1}{c}+\sum_{n=2}^{\infty}\frac{f_{n-1}}{c^n}+\sum_{n=2}^{\infty}\frac{f_{n-2}}{c^n} \\ &= \frac{1}{c}+\frac{1}{c}\sum_{n=2}^{\infty}\frac{f_{n-1}}{c^{n-1}}+\frac{1}{c^2}\sum_{n=2}^{\infty}\frac{f_{n-2}}{c^{n-2}} \\ &= \frac{1}{c}+\frac{1}{c}\sum_{n=1}^{\infty}\frac{f_n}{c^n}+\frac{1}{c^2}\sum_{n=1}^{\infty}\frac{f_n}{c^n} \\ &= \frac{1}{c}+\frac{A}{c}+\frac{A}{c^2} \end{aligned}$$

and solving $A=1/c+A/c+A/c^2$ yields

$$A=\frac{c}{c^2-c-1}.$$

4. Find the area of the set of all points in the unit square, which are closer to the center of the square than to its sides.

Solution:
Let us center the coordinate system at the center of the square, with axes parallel to the sides. The set obviously has a lot of symmetries. Consider the portion in the set $y \geq |x|$, which is obviously a quarter of the whole set. The set may be described analytically by the equations

$$\sqrt{x^2+y^2} \leq \frac{1}{2} - y \quad \text{and} \quad y \geq |x|$$

which leads us to

$$|x| \leq y \leq \frac{1}{4} - x^2.$$

Thus,

$$S = 4\int_{-\frac{\sqrt{2}-1}{2}}^{\frac{\sqrt{2}-1}{2}} (\frac{1}{4} - x^2 - |x|)dx = 8\int_{0}^{\frac{\sqrt{2}-1}{2}} (\frac{1}{4} - x^2 - x)dx = \frac{4\sqrt{2}-5}{3}.$$

5. Let $s = (s_1, \ldots, s_n)$ be a permutation of the numbers $\{1, 2, \ldots, n\}$. A number x is called a *local minimum* of s if x is smaller than both of the numbers on either side of it (or the only number next to it, if x happens to be a_1 or s_n). For example, if $s = (8, 4, 1, 2, 9, 5, 7, 6, 3)$, then the local minima are 1, 5, and 3.

   (1) In terms of n, how many permutations $(s_1, \ldots, s_n)$ are "V-shaped", i.e., have the number 1 as the only local minimum?

   (2) In terms of n, how many permutations $(s_1, \ldots, s_n)$ are "W-shaped", i.e., have 1 and 2 as their local minima, but no others?

   Solution: (1) A V-shaped permutation is specified by which numbers come before 1 and which numbers come after 1 — having made that choice, the earlier numbers must appear in decreasing order and the later ones in increasing order. So the answer is $\boxed{2^{n-1}.}$

   (2) Let s be a W-shaped permutation. Then s has one of the two forms

$$\begin{array}{ccccc} x_1, \ldots, x_p, & 1, & y_1, \ldots, y_q, & 2, & z_1, \ldots, z_r \\ x_1, \ldots, x_p, & 2, & y_1, \ldots, y_q, & 1, & z_1, \ldots, z_r \end{array}$$

   where

   - $X \cup Y \cup Z = B$ as a disjoint union, where $X = \{x_1, \ldots, x_p\}$, $Y = \{y_1, \ldots, y_q\}$, $Z = \{z_1, \ldots, z_r\}$, and $B = \{3, 4, \ldots, n\}$;
   - $q > 0$ (otherwise 2 isn't a local minimum), though p and r can be zero;

- $x_1, \ldots, x_p$ are in decreasing order (so that no x_i is a local min);
- $y_1, \ldots, y_q$ is "$\wedge$-shaped", i.e., has a unique local *maximum*;
- $z_1, \ldots, z_r$ are in increasing order.

By part (a), once we choose which of $1, 2$ comes first and choose the sets X, Y, Z, there are $2^{|Y|-1}$ possibilities for s. For any choice of a nonempty subset Y, there are $2^{n-2-|Y|}$ ways to choose X (and then Z is uniquely determined), so the total number of possibilities for s is

$$\begin{aligned}
2 \sum_{\substack{X \cup Y \cup Z = B\} \\ Y \neq \emptyset}} 2^{|Y|-1} &= 2 \sum_{\emptyset \neq Y \subseteq B} 2^{|Y|-1} 2^{n-2-|Y|} \\
&= 2 \sum_{\emptyset \neq Y \subseteq B} 2^{n-3} \\
&= 2^{n-2} \sum_{\emptyset \neq Y \subseteq B} 1 \\
&= \boxed{2^{n-2}(2^{n-2} - 1).}
\end{aligned}$$