Seventh KS math competition

March 7, 2011

1. Find the real number $a \in (0,1)$ that minimizes the integral

$$I(a) = \int_0^1 |x^n - a^n|\, dx$$

where $n \geq 1$ is an integer.
Solution: Since $f(x) = x^n$ is an increasing function on $[0,1]$ we have

$$I(a) = \int_0^1 |x^n - a^n|\, dx = \int_0^a (a^n - x^n)\, dx + \int_a^1 (x^n - a^n)\, dx,$$

and by direct calculations we get

$$I(a) = \left(a^n x - \frac{x^{n+1}}{n+1}\right)\Bigg|_0^a + \left(\frac{x^{n+1}}{n+1} - a^n x\right)\Bigg|_a^1 = \frac{2n}{n+1}a^{n+1} - a^n + 1.$$

To determine the minimum value of $I(a)$ when $a \in (0,1)$ we observe that

$$\frac{dI(a)}{da} = n(2a-1)a^{n-1},$$

so the only critical point of $I(a)$ in $(0,1)$ is $a = \dfrac{1}{2}$. Since the derivative of $I(a)$ is negative on $\left(0, \dfrac{1}{2}\right)$ and positive on $\left(\dfrac{1}{2}, 1\right)$ we conclude that the value at $a = \dfrac{1}{2}$ is a minimum.

2. Let $x_1, \ldots, x_k$ are disctinct reals. Define polynomials

$$p_k(x) = \prod_{j \neq k} \frac{x - x_j}{x_k - x_j}$$

Show that

$$\sum_{k=1}^{n} p_k(x) = 1,$$

for all x.
Solution: Observe first that $p_k(x_k) = 1, p_k(x_m) = 0$, for all $k, m \neq k$. Thus, for the polynomial

$$P(x) = \sum_{k=1}^{n} p_k(x) - 1,$$

we have $P(x_j) = 0, j = 1, \ldots, n$. Now, $deg(P) = n - 1$ and therefore we have constructed a polynomial of degree $n-1$ with n distinct roots. This is possible only if $P(x) \equiv 0$.

3. Let A be $n \times n$ matrix, such that $A^n = \alpha A$, where α is a real number, different from $+1, -1$. Show that $A \pm I_n$ are invertible.
Solution: Let $B = A - I_n$ ($A + I_n$ is treated similarly). We have $A = B + I_n$ and thus

$$(B + I_n)^n = A^n = \alpha A_n = \alpha(B + I_n)$$

But[1]

$$(B + I_n)^n = I_n + \sum_{k=1}^{n} \binom{n}{k} B^k$$

It follows that

$$\sum_{k=1}^{n} \binom{n}{k} B^k - \alpha B = B(\sum_{k=1}^{n} \binom{n}{k} B^{k-1} - \alpha I_n) = (\alpha - 1)I_n$$

Thus

$$B^{-1} = \sum_{k=1}^{n} \binom{n}{k} B^{k-1} - \alpha I_n.$$

4. Let $n \geq 1$ be an integer and denote by A_n the set of all distinct values of

$$E_n = x_1 + 2x_2 + 3x_3 + \cdots + (n-1)x_{n-1} + nx_n,$$

where each x_i equals either -1 or $+1$ for any $1 \leq i \leq n$.

(i) Determine the sets A_1, A_2, A_3, A_4.

(ii) Find the total number N_n of elements of A_n for each $n \geq 1$.

Answers: $(i) : A_1 = \{-1, 1\}, \quad A_2 = \{-3, -1, 1, 3\}, \quad A_3 = \{-6, -4, -2, 0, 2, 4, 6\},$

$$A_4 = \{-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10\}.$$

$$(ii) : N_n = 1 + \frac{n(n+1)}{2}.$$

Solutions: Since $E_{n+1} = E_n + (n+1)x_{n+1}$ and x_{n+1} equals -1 or $+1$ we get that for any $n \geq 1$ the set A_{n+1} is the union of two shifts of A_n,

$$(*) \quad A_{n+1} = \{a - (n+1) \mid a \in A_n\} \cup \{a + (n+1) \mid a \in A_n\}, \quad n \geq 1.$$

(i) Equation $(*)$ can now be used to first determine A_2 from $A_1 = \{-1, 1\}$ by adding -2 or 2 to the elements of A_1, then to determine A_3 from A_2 by adding -3 or 3 to the elements of A_2, and finally to get A_4 from A_3 by adding -4 or 4 to the elements of A_3.

[1] here we do not need to know the precise coefficients of the polynomials

(ii) Suppose next that the elements of A_n are listed in ascending order,

$$(**) \quad a_1 < a_2 < \cdots < a_{N-1} < a_N, \quad N = N_n.$$

Obviously,

$$a_1 = -(1+2+3+\cdots+n) = -\frac{n(n+1)}{2}, \quad a_N = 1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

The sets A_n are completely described by the following

Lemma. *Sequence $(**)$ is an arithmetic progression with common difference $d = 2$.*

We can prove this result by induction on n, using equation $(*)$ and based on two observations:

• *any shift of an arithmetic progression is an arithmetic progression;*

and

• *the union of two arithmetic progressions with the same common difference and at least one common term is an arithmetic progression.*

Here is a different proof that does not use induction! We claim that

(a) *For any $a, b \in A_n$, the difference $b - a$ is an even number;* and

(b) *If $a \in A_n$ and $a < a_N$, then $a + 2 \in A_n$,*

two properties which show that $(**)$ is an arithmetic progression with $d = 2$.

To prove (a), suppose $a = x_1 + 2x_2 + \ldots nx_n$ and $b = y_1 + 2y_2 + \ldots ny_n$. Since $y_i - x_i$ equals -2, 0, or 2, for each $1 \le i \le n$, the difference $b - a$ must be even.

To prove (b), suppose $a = x_1 + 2x_2 + \ldots nx_n$ with $a < a_N$ and analyze the following two situations, $x_1 = -1$ or $x_1 = 1$.

If $x_1 = -1$, define $b \in A_n$ as $b = y_1 + 2y_2 + \ldots ny_n$, where $y_1 = 1$ and $y_i = x_i$ for $i \ge 2$. Obviously, $b - a = 2$, so $a + 2 = b \in A_n$.

If $x_1 = 1$, there exists $k \ge 2$ such that $x_{k-1} = 1$ and $x_k = -1$. Define $b \in A_n$ as $b = y_1 + 2y_2 + \ldots ny_n$ with $y_{k-1} = -1$, $y_k = 1$, and $y_i = x_i$ for all the other values of i. Once more we get $b - a = 2$, so $a + 2 = b \in A_n$. The proof is complete.

It remains to find $N = N_n$, the total number of elements of A_n. From our Lemma we know that $a_N = a_1 + (N-1)d$, that is,

$$\frac{n(n+1)}{2} = -\frac{n(n+1)}{2} + 2(N-1),$$

whence $N = 1 + \dfrac{n(n+1)}{2}$ for each $n \ge 1$.

5. Take an 8×8 chessboard and delete one white square and one black square at random, leaving a shape with 62 squares. Is it always possible to cover the remaining 62 squares with 31 dominoes, with each domino covering two adjacent squares, no matter which two squares were initially deleted?
Answer: Yes.