Eight KS math competition - solutions

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1. Find the largest value of the expression $a^2 + b^2 + c^2$, if it is known that the quadratic function

$$f(x) = ax^2 + bx + c,$$

satisfies $|f(x)| \leq |x|$ for all $x : -1 \leq x \leq 1$.

Solution: Since $|c| = |f(0)| \leq |0| = 0$, it follows $c = 0$ and we have to look at $f(x) = ax^2 + bx$. Thus

$$|f(x)| = |x||ax + b| \leq |x|, \tag{1}$$

whence $|ax + b| \leq 1$. At the end-points $x = \pm 1$, we have

$$|a + b| \leq 1, |-a + b| \leq 1.$$

It follows that

$$a^2 + b^2 = \frac{1}{2}\left((-a + b)^2 + (a + b)^2\right) \leq 1.$$

Hence

$$a^2 + b^2 + c^2 = a^2 + b^2 \leq 1.$$

Equality is achieved, when $|a+b| = 1$ and $|-a+b| = 1$, which happens exactly when $(a, b) = (\pm 1, 0)$ or $(a, b) = (0, \pm 1)$.

2. Compute

$$\int_0^\infty \frac{\arctan \pi x - \arctan x}{x}\, dx$$

Solution: Let $N >> 1$. Consider

$$f_N(a) = \int_0^N \frac{\arctan(ax)}{x} dx.$$

Taking a derivative in a yields

$$f_N'(a) = \int_0^N \frac{1}{1 + a^2x^2} dx = \frac{1}{a} \int_0^{aN} \frac{1}{1 + y^2} dy = \frac{1}{a} \tan^{-1}(aN)$$

The integral in question is

$$\lim_{N\to\infty} [f_N(\pi) - f_N(1)] = \lim_{N\to\infty} \int_1^\pi \frac{1}{a} \tan^{-1}(aN) da =$$

$$= \int_1^\pi \frac{1}{a} \lim_{N\to\infty} \tan^{-1}(aN) da = \frac{\pi}{2} \int_1^\pi \frac{1}{a} da = \frac{\pi}{2} \ln(\pi).$$

3. Is the number of solutions $(x_1, x_2, x_3) \in \mathbb{N}^3$ to $\frac{2^8}{x_1} + \frac{2^8}{x_2} + \frac{2^8}{x_3} = 1$ even or odd?
Solution: The solutions where $x_2 \neq x_3$ come in pairs so they can be excluded. Hence, it is equivalent to count the number of solutions where $x_2 = x_3$, which is the number of solutions $(x_1, x_2) \in \mathbb{N}^2$ of $\frac{2^8}{x_1} + \frac{2^9}{x_2} = 1$. For $(x_1, x_2) \in \mathbb{N}^2$, $\frac{2^8}{x_1} + \frac{2^9}{x_2} = 1$ if and only if $2^8 x_2 + 2^9 x_1 = x_1 x_2$, which can be rewritten as $(x_1 - 2^8)(x_2 - 2^9) = 2^{17}$. It follows that the solutions $(x_1, x_2) \in \mathbb{N}^2$ to $\frac{2^8}{x_1} + \frac{2^9}{x_2} = 1$ are of the form $(x_1, x_2) = (2^8 + 2^a, 2^9 + 2^{17-a})$ for $a \in \{0, 1, \dots, 17\}$. Thus, the number of solutions $(x_1, x_2, x_3) \in \mathbb{N}^3$ to $\frac{2^8}{x_1} + \frac{2^8}{x_2} + \frac{2^8}{x_3} = 1$ is even.

4. The sequence $\{a_n\}_n$ is defined via $a_1 = 0$, $a_{n+1} = \sqrt{20 - a_n}$. Prove that a_n is convergent and find its limit.
Solution: Compute $a_2 = \sqrt{20} \sim 4.47$, $a_3 = \sqrt{20 - \sqrt{20}} \sim 3.94$, $a_4 = \sqrt{20 - \sqrt{20 - \sqrt{20}}} \sim 4.007$ etc. We show by induction that $a_{2k+2} < a_{2k}, a_{2k+2} > 4$, while $a_{2k+1} > a_{2k-1}, a_{2k+1} < 4$. Indeed, the checking is done for $k = 0$. Assuming $a_{2k} > a_{2k-2}$, we have

$$a_{2k+2} = \sqrt{20 - a_{2k+1}} = \sqrt{20 - \sqrt{20 - a_{2k}}} > \sqrt{20 - \sqrt{20 - a_{2k-2}}} = a_{2k}.$$

Similarly for the proof of $a_{2k+1} > a_{2k-1}$. If $a_{2k-1} < 4$, we have

$$a_{2k} = \sqrt{20 - a_{2k-1}} > \sqrt{16} = 4.$$

Similarly for $a_{2k+1} < 4$, based on $a_{2k} > 4$. Thus $\{a_{2k}\}_k, \{a_{2k+1}\}_k$ are convergent. Denote their respective limits by $a = \lim_k a_{2k} \geq 4, b = \lim_k a_{2k+1} \leq 4$. We have, after passing to limits

$$a = \sqrt{20 - b}, \; b = \sqrt{20 - a}.$$

In order to solve the system, we have

$$a^2 = 20 - b = 20 - \sqrt{20 - a},$$

whence

$$16 - a^2 = \sqrt{20 - a} - 4 = \frac{4 - a}{\sqrt{20 - a} + 4}.$$

The last one is equivalent to

$$(4 - a)\left(4 + a - \frac{1}{\sqrt{20 - a} + 4}\right) = 0$$

The only positive solution to the last one is $a = 4$, since

$$4 + a - \frac{1}{\sqrt{20 - a} + 4} \geq 4 - \frac{1}{4} > 0.$$

5. We are given $n \geq 3$ points in the plane. Prove that there are three of them that form an angle of at most π/n.

Solution: If any three points are co-linear, then we are done. So, assume that no three points are on the same line. Then, take the convex hull of the n points and take three consecutive points on it, say P_1, P_2, P_n (so that P_1 is in the middle). Now

$$\pi = \angle P_2P_1P_n + \angle P_1P_nP_2 + \angle P_1P_2P_n$$

Observe that now, because of the construction (all remaining points are inside the angle $P_1P_2P_n$)

$$\angle P_2P_1P_n = \sum_{k=2}^{n-1} P_kP_2P_{k+1}$$

Alltogether,

$$\pi = \angle P_2P_1P_n + \angle P_1P_nP_2 + \angle P_1P_2P_n = \angle P_1P_nP_2 + \angle P_1P_2P_n + \sum_{k=2}^{n-1} P_kP_2P_{k+1}$$

Thus, at least one of the n angles in the last identity will be less than or equal to π/n.