2015 Kansas MAA Undergraduate Mathematics Competition: Solutions!

1. Show that the equation $x^2 + 2015x + 2016y^3 = 3$ has no integer solutions $x, y \in \mathbb{Z}$.

   Solution: Given any integers $x, y \in \mathbb{Z}$, it is clear that $2016y^3$ will be an even integer. Further, if x is even then $x^2 + 2015x$ is the sum of two even integers, and hence even, while if x is odd then $x^2 + 2015x$ is the sum of two odd numbers, and hence also even. In any case, the sum $x^2 + 2015x + 2016y^3$ will be even for any $x, y \in \mathbb{Z}$, and hence the given equation can have no integer solutions.

2. Find the first digit (the ones digit) in the sum

   $$1! + 2! + 3! + \cdots + 2015!$$

   Solution: Notice that the ones digit in the above sum is given by the sum mod 10. Since $n! = 0 \mod 10$ for all $n \geq 5$, it follows that

   $$1! + 2! + 3! + \cdots + 2015! \mod 10 = 1! + 2! + 3! + 4! \mod 10 = 3.$$

   Thus, the ones digit of the above sum is 3.

3. Determine whether there exists an infinite sequence (a_n) of positive real numbers such that the series

   $$\sum_{n=1}^{\infty} \frac{a_1 + a_2 + \ldots + a_n}{n}$$

   converges.

   Solution: Since all of the a_n are positive, it follows that, for each $n \in \mathbb{N}$,

   $$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \frac{a_1}{n}$$

   and hence the series diverges by the comparison test.

4. Let $f(x)$ be a strictly positive continuous function. Evaluate the integral

   $$\int_0^4 \frac{f(x)}{f(x) + f(4-x)}\, dx.$$

   Solution: Letting I denote the value of the above integral, we have

   $$I = \int_0^4 \frac{f(x) + f(4-x) - f(4-x)}{f(x) + f(4-x)}\, dx = 4 - \int_0^4 \frac{f(4-x)}{f(x) + f(4-x)}\, dx.$$

Making the substitution $y = 4 - x$ we find

$$\int_0^4 \frac{f(4-x)}{f(x)+f(4-x)}\,dx = -\int_4^0 \frac{f(y)}{f(4-y)+f(y)}\,dy = I$$

so that, combining with the previous identity, we find

$$I = 4 - I \Rightarrow I = 2.$$

5. Suppose that $f, g : \mathbb{N} \to \mathbb{N}$, f is onto, g is one-to-one, and $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. Prove that $f(n) = g(n)$ for all $n \in \mathbb{N}$.

   Solution 1: Since f is onto, there exists a $n_1 \in \mathbb{N}$ such that $f(n_1) = 1$. Since $g(n_1) \leq f(n_1)$ and $g(n_1) \in \mathbb{N}$, it follows that $g(n_1) = 1 = f(n_1)$ Furthermore, since g is one-to-one, it follows that n_1 is the unique natural number with this property. Similarly, since f is onto there exists an $n_2 \in \mathbb{N}$ such that $f(n_2) = 2$. Since $g(n_2) \leq f(n_2)$ it follows that $g(n_2) \in \{1, 2\}$. Using that g is one-to-one, it follows that $g(n_2) \neq 1$ (since $n_1 \neq n_2$) and hence that $g(n_2) = 2$. Thus, $g(n_2) = f(n_2) = 2$ and, again by the fact that g is one-to-one, n_2 is the unique natural number with this property. Continuing by induction, it follows that, for each $k \in \mathbb{N}$ there exists a unique $n_k \in \mathbb{N}$ such that $f(n_k) = g(n_k) = k$.

   It now remains to show that $\{n_k\}_{k=1}^{\infty} = \mathbb{N}$, but this is clear since if $m \in \mathbb{N} \setminus \{n_k\}_{k=1}^{\infty}$ then $g(m) \in \mathbb{N}$ and hence, by above, there exists a unique $k \in \mathbb{N}$ such that $g(n_k) = f(n_k) = g(m)$. Since g is one-to-one, it follows that $m = n_k$ and hence that $\{n_k\}_{k=1}^{\infty} = \mathbb{N}$, as claimed. Together then, it follows that $f(n) = g(n)$ for every $n \in \mathbb{N}$.

   Solution 2: By contradiction. Assume there exists $m_0 \in \mathbb{N}$ with $g(m_0) < f(m_0)$. Since f is onto, there exists $m_1 \in \mathbb{N}$ with $f(m_1) = g(m_0)$. Since g is one-to-one, and $g(m_0) = f(m_1)$, $g(m_1) < f(m_1) = g(m_0)$.

   Now assume that there exists positive integers m_0, m_1, $\ldots m_k$ with $g(m_k) < g(m_{k-1}) < \cdots < g(m_0)$. Since f is onto, there exists $m_{k+1} \in \mathbb{N}$ with $f(m_{k+1}) = g(m_k)$. Since g is one-to-one, $g(m_{k+1}) < f(m_{k+1}) = g(m_k)$.

   By induction, we obtain a sequence of positive integers $\{m_k\}$ with the property that $\{g(m_k)\}$ is a strictly decreasing sequence. Since $\{g(m_k)\} \subset \mathbb{N}$, this is a contradiction.

6. For each $n \in \mathbb{N}$, show that

$$\sum_{k=0}^{n} \sum_{i=0}^{n-k} \binom{n}{k} \binom{n-k}{i} 3^{k+i} = 7^n.$$

Solution: First, rewrite the sum as

$$\sum_{k=0}^{n} \binom{n}{k} 3^k \sum_{i=0}^{n-k} \binom{n-k}{i} 3^i.$$

By the binomial theorem, we know

$$\sum_{i=0}^{n-k} \binom{n-k}{i} 3^i = (1+3)^{n-k} = 4^{n-k}$$

and hence, using the binomial theorem again, the given sum is equal to

$$\sum_{k=0}^{n} \binom{n}{k} 3^k 4^{n-k} = (4+3)^n = 7^n,$$

as claimed.

7. Let $f : [0,1] \to \mathbb{R}$ be an integrable function (not necessarily continuous). Prove that if $f(t) \le 2$ for all $t \in [0,1]$ then there exists a unique solution $x \in [0,1]$ of the equation

$$3x - 1 = \int_0^x f(t)dt.$$

Solution: Define the function $g : [0,1] \to \mathbb{R}$ by $g(x) = 3x - 1 - \int_0^x f(t)dt$. Since f is integrable, it follows by the fundamental theorem of calculus that g is continuous on $[0,1]$. Now, notice that $g(0) = -1$ and that

$$g(1) = 2 - \int_0^1 f(t)dt \ge 2 - \int_0^1 2\ dt = 0.$$

If $g(1) = 0$, then $x = 1$ is a solution to the equation. If $g(1) > 0$, by the intermediate value theorem, we have that there exists some $c \in (0,1)$ such that $g(c) = 0$. To see that the solution is unique, we claim that g is a strictly increasing function on $[0,1]$. Indeed, given any $a, b \in [0,1]$ with $a < b$ we have

$$g(b) - g(a) = 3(b-a) - \int_a^b f(t)dt \ge 3(b-a) - \int_a^b 2\ dt = b - a > 0.$$

Thus, g is strictly increasing on $[0,1]$ and hence c is unique, as claimed.

8. Suppose $f : [0,\infty) \to \mathbb{R}$ is a continuous, non-negative function. Suppose that $f(x+1) = (1/2)f(x)$ for all $x \ge 0$, and $\int_0^1 f(x)\,dx = 100$. Show that the integral $\int_0^\infty f(x)\,dx$ exists and find its value.

Solution: First, we claim that the given improper integral converges. Indeed, note that since f is non-negative it follows that the function $g : \mathbb{R} \to [0, \infty)$ given by

$$g(t) = \int_0^t f(x)dx$$

is a non-decreasing function and, furthermore, given any $t \geq 1$ we have implies

$$g(t) \leq \int_0^{\lceil t \rceil} f(x)dt = \sum_{n=0}^{\lceil t \rceil} \int_0^1 f(x+n)dx$$
$$= \sum_{n=0}^{\lceil t \rceil} (1/2)^n \int_0^1 f(x)dx \leq 100 \sum_{n=0}^{\infty} (1/2)^n = 200.$$

Thus, g is a monotone non-decreasing function that is bounded above. It follows that $\lim_{t\to\infty} g(t)$ exists, which proves the given improper integral converges, as claimed.

Now, to evaluate the integral, it sufficies to calculate the limit $\lim_{t\to\infty} g(t)$. Since we know the limit exist, we can take the limit along the natural numbers. To this end, for each $n \in \mathbb{N}$ notice that

$$g(n) = \sum_{k=0}^{n} \int_0^1 f(x+k)dx = \sum_{k=0}^{n} (1/2)^k \int_0^1 f(x)dx = 100 \sum_{k=0}^{n} (1/2)^k$$

and hence that

$$\lim_{n\to\infty} g(n) = 100 \sum_{k=0}^{\infty} (1/2)^k = 200.$$

It follows that

$$\int_0^{\infty} f(x)dx = 200.$$

9. Determine, with proof, all polynomials satisfying satisfying $P(0) = 0$ and $P(x^2 + 1) = (P(x))^2 + 1$ for all x.

Solution: We claim the only polynomial satisfying the given properties is $P(x) = x$. To see this, first notice that the condition $P(0) = 0$ implies

$$P(0^2 + 1) = P(0)^2 + 1 = 1.$$

so that $P(1) = 1$. Similarly, we find

$$P(1^2 + 1) = P(1)^2 + 1 = 1^2 + 1$$
$$P((1^2 + 1)^2 + 1) = P(1^2 + 1)^2 + 1 = (1^2 + 1)^2 + 1.$$

By induction, it follows that if we define the recursive sequence

$$a_1 = 1, \quad a_{n+1} = a_n^2 + 1 \quad \text{for all} \;\; n \in \mathbb{N}$$

then $P(a_n) = a_n$ for all $n \in \mathbb{N}$. Furthermore, the sequence $\{a_n\}_{n=1}^{\infty}$ is strictly increasing since for all $n \in \mathbb{N}$ we have

$$a_{n+1} - a_n = a_n^2 + 1 - a_n = (a_n - 1/2)^2 + 3/4 > 0.$$

It follows that the function $G(x) = P(x) - x$ is a polynomial with infinitely many distinct real roots. Since a non-trivial polynomial of degree m can have at most m real roots by the fundamental theorem of algebra, it follows that $G(x) = 0$ for all $x \in \mathbb{R}$, i.e. $P(x) = x$ for all $x \in \mathbb{R}$, as claimed.

10. Suppose that you have a $2^n \times 2^n$ grid with a single 1×1 square removed. Prove that the remaining squares can be tiled with L-shaped tiles consisting of three 1×1 tiles - that is, a 2×2 tile with a single 1×1 square removed.

    Solution: We prove this by induction on n. Clearly, this is possible if $n = 1$, since a 2×2 grid with exactly one square removed is precisely the shape of a given L-shaped tile. Now, suppose that, for some given $n \in \mathbb{N}$ it is known that that this is possible for any given $2^n \times 2^n$ grid with exactly one square removed can be covered by such L-shaped tiles. Consider then a $2^{n+1} \times 2^{n+1}$ grid and notice this can be decomposed into a 4 $2^n \times 2^n$ sub-grids in a unique way. Furthermore, we can consider the middle 2×2 sub-grid of the $2^{n+1} \times 2^{n+1}$ grid made up of exactly one square from each of the four $2^n \times 2^n$ sub-grids. Since there is exactly one square missing from the given $2^{n+1} \times 2^{n+1}$ grid, it follows that exactly one of these $2^n \times 2^n$ sub-grids has exactly one square missing (by hypothesis) and hence can be tiled by such L-shaped tiles by the induction hypothesis. After covering this $2^n \times 2^n$ sub-grid, it follows that the center 2×2 grid has exactly one square missing, and hence can be tiled with exactly one L-shaped tile. The remaining 3 $2^n \times 2^n$ sub-grids now each have exactly one square covered, and hence can each be tiled by the given L-shaped tiles by the induction hypothesis. Thus, the given $2^{n+1} \times 2^{n+1}$ grid can be tiled by such L-shaped tiles. The proof is thus complete by mathematical induction.