2016 Kansas MAA Undergraduate Mathematics Competition and Solutions

1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Compute $\det\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}A^{2k}\right)$.

   Solution: Simply observe that

   $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}A^{2k} = (I - A^2)^n$$

   so that

   $$\det\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}A^{2k}\right) = n\det\left(I - A^2\right) = n\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -3^n.$$

2. Two points P and Q are randomly selected in the interval $[0, 2]$. What is the probability that P and Q are within a distance of $1/3$ from each other, i.e. determine

   $$\text{Prob}\left(\text{dist}(P, Q) \leq \frac{1}{3}\right).$$

   Solution: In the square $[0, 2]^2$, the area between the curves $P = Q + 1/3$ and $P = Q - 1/3$ is $4 - (5/3)^2 = 11/9$. Dividing by 4 gives the probability being $11/36$.

3. Evaluate the integral

   $$\int_0^4 \left(x^2 - 4x + 7\right)\sin\left(x^3 - 6x^2 + 12x - 8\right)dx.$$

   Solution: By either recognizing that $x^3 - 6x^2 + 12x - 8 = (x - 2)^3$ directly, or coming to this conclusion by noting that $x = 2$ is the unique critical point (and root), this motivates the change of variables $y = x - 2$, which transforms the integral into

   $$\int_{-2}^{2} \left(y^2 + 3\right)\sin(y^3)dy.$$

   The above integrand is odd and hence integrates to zero.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(f(x)) = x$ has exactly 2016 solutions. Show that f must have an even number of fixed points, i.e. solutions of $f(x) = x$.

   Solution: Note that if $f(f(x)) = x$, then setting $y = f(x)$ we see that $f(f(y)) = y$ as well. Thus, there must be an even number of solutions of $f(f(x)) = x$ that are not themselves fixed points of x. Since 2016 is even, it follows that f must have an even number of fixed points as well.

5. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is an increasing function that is additive, i.e. $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}$, and satisfies $\lim_{x\to\infty} f(x)=\infty$. Prove that the limit

$$\lim_{x\to\infty}\frac{[f(x)]}{f([x])}$$

exists, and determine its value. Here, $[y]$ denotes the integer part of a given $y\in\mathbb{R}$. That is, $[y]$ is the largest integer less than or equal to y.

Solution: By definition of $[\cdot]$ and the monotonicity of f, we have $f(x)-1\leq[f(x)]\leq f(x)$ and $f(x)-f(1)\leq f([x])\leq f(x)$ so that

$$\frac{f(x)-1}{f(x)}\leq\frac{[f(x)]}{f([x])}\leq\frac{f(x)}{f(x)-f(1)}$$

for all $x\in\mathbb{R}$. By the squeeze theorem, it follows that the given limit exists and equals 1.

6. A bicyclist completes a 12 mile ride in 60 minutes. Prove that there exists a continuous 3-mile segment within this 12 miles that the rider completed in exactly 15 minutes.

Solution: For each $0\leq x\leq 9$, let $T(x)$ denote the amount of time it took the rider to ride between x and $x+3$ miles. Clearly $T(x)$ is continuous along the course and

$$T(0)+T(3)+T(6)+T(9)=60,$$

which clearly implies that not all of $T(0)$, $T(3)$, $T(6)$, and $T(9)$ can be less than 15, and not all can be greater than 15. Thus, there exists integers m,n with $0\leq m,n\leq 9$ such that

$$T(m)\leq 15\leq T(n).$$

By the intermediate value theorem, it follows that there exists some time $t^*\in[\min(m,n),\max(m,n)]$ such that $T(t^*)=15$, as claimed.

7. Given some positive integer $p\geq 1$, let 2^p-1 be a prime number and set $n=2^{p-1}(2^p-1)$. Show that the sum of all the the positive integer divisors of n (not including n itself) is equal to $2n$.

Solution: Set $q=2^p-1$ and note that since q is prime the divisors of n,not including n itself, are

$$1,\ 2,\ 2^2,\ldots,\ 2^{p-1}\quad\text{and}\quad q,\ 2q,\ 2^2q,\ldots,\ 2^{p-2}q$$

Summing the first collection of divisors gives

$$1+2+2^2+\ldots+2^{p-1}=\frac{2^p-1}{2-1}=2^p-1=q$$

while the sum of the other collection of divisors gives

$$q\left(1+2+2^2+\ldots+2^{p-2}\right)=q\left(\frac{2^{p-1}-1}{2-1}\right)=2^{p-1}q-1=n-q.$$

Therefore, the sum of all such divisors is precisely

$$q+n-q=n,$$

as claimed.

8. Find all real solutions to the system

$$\begin{cases} x^3+y^3+z^3=0 \\ x^5+y^5+z^5=0 \\ x^7+y^7+z^7=0. \end{cases}$$

Solution: Clearly $(x,y,z)=(0,0,0)$ works. If $(x,y,z)\in\mathbb{R}^3$ is a nontrivial solution of the given system, then if follows that the vector $(1,1,1)\in\mathbb{R}^3$ is orthogonal to the three vectors (x^3,y^3,z^3), (x^5,y^5,z^5), and (x^7,y^7,z^7), and hence these latter three vectors must be linearly dependent, i.e.

$$\begin{aligned} 0=\det\begin{pmatrix} x^3 & y^3 & z^3 \\ x^5 & y^5 & z^5 \\ x^7 & y^7 & z^7 \end{pmatrix} &= (xyz)^3\det\begin{pmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{pmatrix} \\ &= -(xyz)^3(x^2-y^2)(x^2-z^2)(y^2-z^2). \end{aligned}$$

It follows that either $x^2=y^2$, $x^2=z^2$, or else $y^2=z^2$.

Now, notice that if any two of x,y,z are the same, then $x=y=z=0$. Indeed, if $x=y$, for example, then the first and second of the given equations imply that $z=2^{1/3}x=2^{1/5}x$, and hence that $x=0$. Furthermore, it is clear that if, for example, $x=-y$, then $z=0$. It follows that the solutions of the given system are given by

$$\{(a,-a,0),(a,0,-a),(0,a,-a):a\in\mathbb{R}\}.$$

9. Suppose that $P(x)$ is a polynomial with integer coefficients that takes the value 1 at three distinct integers. Prove that $P(x)$ can not have an integer root.

Solution: Suppose that $P(r)=0$ for some integer r, and let a be an integer with $P(a)=1$. Then clearly $P(a)-P(r)=1$ and hence, since P has integer coefficients, it follows that the integer $a-r$ divides 1. But then $a-r=\pm 1$ which only gives two possibilities for the root a, not three. Thus, no such polynomial can exist.

10. Let $n \geq 2$ be a fixed integer and $a > 0$. Determine all functions $f(x)$ that are bounded on $0 < x < a$ and which satisfy the functional equation

$$f(x) = \frac{1}{n^2}\left(f\left(\frac{x}{n}\right) + f\left(\frac{x+a}{n}\right) + \ldots + f\left(\frac{x+(n-1)a}{n}\right)\right)$$

for all $x \in (0, a)$.

Solution: First, since f is bounded there exists a $M > 0$ such that $|f(x)| \leq M$ for all $x \in (0, a)$. Next, notice that for every $k = 0, 1, 2, \ldots (n-1)$ and $x \in (0, a)$ we have

$$0 < \frac{x+ka}{n} < a$$

and hence

$$\left|f\left(\frac{x+ka}{n}\right)\right| \leq M$$

for every $k = 0, 1, 2, \ldots, n-1$. From the functional equation, it follows that

$$|f(x)| \leq \frac{1}{n^2}(K + K + K + \ldots + K) \leq \frac{K}{n}$$

for all $x \in (0, a)$, effectively improving our upper bound by a factor of $\frac{1}{n}$. From the functional equation again, we find

$$|f(x)| \leq \frac{1}{n^2}\left(\frac{K}{n} + \frac{K}{n} + \frac{K}{n} + \ldots + \frac{K}{n}\right) \leq \frac{K}{n^2}$$

for all $x \in (0, a)$. Continuing in this way, we find that, for every $j = 0, 1, 2, \ldots$ we have the uniform bound $|f(x)| \leq \frac{K}{n^j}$ valid for all $x \in (0, a)$. Taking $j \to \infty$ implies that $f(x) = 0$ for all x.