2017 Kansas MAA Undergraduate Mathematics Competition

Instructions:

1. This is a *team competition*. You are permitted to work with the members of your team on the following 10 problems. You are *not* permitted to ask for or receive any assistance from anyone other than your team mates.

2. Each team should submit at most one solution to each problem. Each solution should be completed on a separate sheet of paper. Also, each solution page should the have problem number AND the teams **number** written on it (your team number is located in the upper right hand corner of this paper). Please do not write your school name, the names of the team mates, or any other identifying information on the solutions.

3. You have until 11am to complete the exam. When you are finished with the exam, each team should place the problem sheet and each solution page (with the team number and problem number written on it) into the exam envelope. Any scrap or extra paper should be returned to the graders.

4. Calculators are permitted. However, you must show all of your work for full credit.

5. No cell phones or other electronic devices (other than calculators) are permitted during the exam.

Problems:

1. Find a closed form (your final answer should not use the summand Σ notation) for the probability of getting an even number of 1s in a toss of n fair die.

2. Let $b > 1$ and suppose that $x = 101101101 \cdots 101$ in base b. Here, x is supposed to have $3n$ digits for some integer $n > 0$. Assuming $x > 101$ in base b, prove that x is not prime.

3. A circle rolls along the inside arc of the parabola $y = x^2$. What is the radius of the largest circle that will eventually reach the bottom of the parabola (its absolute min) without getting stuck before getting there?

4. Let d be a positive integer and suppose $f : [0, d] \to \mathbb{R}$ is a continuous function with $f(0) = f(d)$. Show that $f(x) = f(x + 1)$ for some $x \in [0, d - 1]$.

Exam continued on back!

5. Let S be a finite set of positive integers, and define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n \in S} \frac{\sin(e^{nx} - 1)}{n}$$

Suppose that $|f(x)| \leq 2017|x|$ for all x. Prove that S can have at most 2017 elements.

6. Let A be a $n \times n$ matrix such that every entry of A is either 1 or -1. Clearly $\det(A)$ is an integer. Show that $\det(A)$ is divisible by 2^{n-1}.

7. Characterize the positive integers n for which n^{500} ends in the digits 9376.

   Hint: *Given $n \in \mathbb{N}$, $\phi(n)$ denotes Euler's totient function: the number of positive integers less than or equal to n which are relatively prime to n. Feel free to use the following facts, if you find them helpful:*

   Fact 1 $a^{\phi(n)} \equiv 1 \mod n$ *for relatively prime* $a, n \in \mathbb{N}$.

   Fact 2 $\phi(p^n) = p^n - p^{n-1}$ *for all prime* p *and* $n \in \mathbb{N}$.

8. Define a sequence of integers as follows: for each $n \in \mathbb{N}$, let m_n be the largest integer so that

$$m_n > \frac{m_n^2}{5n} + \frac{4n}{5} + \frac{m_n}{5n} - \frac{2}{5}.$$

   Here, $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the "natural numbers". First show that such an m_n exists, for each $n \in \mathbb{N}$, then show that the limit below exists and calculate it:

$$\lim_{n \to \infty} \frac{m_n}{n}.$$

9. Determine, with proof, all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the following properties:

   (i) $f(3) = 3$.

   (ii) $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

   (iii) $f(m) > f(n)$ whenever $m > n$.

10. Please prove or disprove: there exists a subset A of 2000 elements of $X = \{1, 2, \ldots, 3000\}$ in which no member of A is twice another member of A.