2019 Kansas MAA Undergraduate Mathematics Competition

Instructions:

1. This is a *team competition*. You are permitted to work with the members of your team on the following 10 problems. You are not permitted to ask for or receive any assistance from anyone other than your team mates.

2. Each team should submit at most one solution to each problem. Each solution should be completed on a separate sheet of paper. Also, each solution page should the have problem number **AND** the teams number written on it (your team number is located in the upper right hand corner of this paper). Please do not write your school name, the names of the team mates, or any other identifying information on the solutions.

3. You have until 11am to complete the exam. When you are finished with the exam, each team should place the problem sheet and each solution page (with the team number and problem number written on it) into the exam envelope. Any scrap or extra paper should be returned to the graders.

4. Calculators are permitted. However, you must show all of your work for full credit.

5. No cell phones or other electronic devices (other than calculators) are permitted during the exam.

Problems:

1. Suppose a dresser with 15 drawers contains 104 coins. Prove or disprove: there is a pair of drawers that contain the same number of coins (some drawer could have no coins).

   Solution: There must be such a pair of drawers. If each drawer had a distinct amount of coins, the least amount of coins we could have would be $1 + 2 + \ldots + 14 = 105$.

2. Suppose $f(x, y)$ is a real-valed function with domain $\mathbb{R}^2$. Suppose the sum of the function's values taken over the vertices of any equilateral triangle is 0. That is, given any 3 points $P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3) \in \mathbb{R}^2$ that are the vertices of an equilateral triangle,

   $$f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3) = 0.$$

   Prove or disprove: $f(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$.

   Solution: The function f must indeed be identically 0. It suffices to show that the function is constant. Let P_1, P_2 be two arbitrary points of $\mathbb{R}^2$. Construct a parallelogram with 2 $60°$-angles, and 2 $120°$-angles, so that P_1, P_2

are the two vertices at the 60° angles. Let P_3, P_4 be the remaining two vertices. Because $\{P_1, P_3, P_4\}$ and $\{P_2, P_3, P_4\}$ both define equilateral triangles, we infer

$$f(P_1) + f(P_3) + f(P_4) = 0$$
$$f(P_2) + f(P_3) + f(P_4) = 0.$$

Hence, $f(P_1) = f(P_2)$, which implies f is constant as P_1, P_2 were arbitrary. The only constant function satisfying the summation condition is clearly the zero function.

3. Given an integer $n \geq 2$, determine (with proof) whether or not the equation

$$\sin(x)\sin(2x)\cdots\sin(nx) = 1$$

has a real solution $x \in \mathbb{R}$ or not.

Solution: There exists no solution. Observe that if x solves the above then we must have $|\sin(jx)| = 1$ for all $j = 1, 2, \ldots, n$. Since $n \geq 2$, this means at the very least that

$$|\sin(x)| = 1 \text{ and } |\sin(2x)| = 1.$$

The first of these conditions requires that $x = \frac{\pi}{2} + k\pi$ for some integer k, while the second requires that $2x = \frac{\pi}{2} + m\pi$ for some integer m. Eliminating x implies that

$$\pi + 2k\pi = \frac{\pi}{2} + m\pi,$$

or, equivalently,

$$m - 2k = \frac{1}{2}.$$

which is impossible since m and k are integers. Therefore, no solution exists.

4. Steven, Rachel, and Alexis take turns rolling a fair, 6-sided die. Steven begins, Rachel follows Steven, Alexis follows Rachel, and this order repeats. The game ends when a person rolls a 6, and this person is declared the winner. Find the probability that Alexis wins. Your final answer should be a fraction with numerator and denominator both integers!

Solution: The probability Alexis gets the first 6 in round 1 is

$$\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6},$$

in round 2 is

$$\left(\frac{5}{6}\right)^5 \cdot \frac{1}{6},$$

and in generally in round k is

$$\left(\frac{5}{6}\right)^{3k-1} \cdot \frac{1}{6}.$$

So the desired probability is given by

$$\begin{aligned}
\sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{3k-1} \cdot \frac{1}{6} &= \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{3k} \\
&= \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{125}{216}\right)^{k} \\
&= \frac{1}{5} \left(\frac{1}{1 - \frac{125}{216}} - 1\right) \\
&= \frac{1}{5} \left(\frac{216}{91} - 1\right) \\
&= \frac{1}{5} \left(\frac{125}{91}\right) \\
&= \frac{25}{91}
\end{aligned}$$

5. Define the sequence

$$a_n = \sum_{k=1}^{n} \frac{1}{4^k}.$$

Determine, with proof, whether the series $\sum_{n=1}^{\infty} (a_n)^{2n}$ converges.

Solution For each n, a_n is the nth partial sum of a convergent geometric series. In fact,

$$a_n \leq \frac{1}{1 - \frac{1}{4}} - 1 = \frac{1}{3}$$

for all n. So, $(a_n)^{2n} \leq \frac{1}{9^n}$ and hence the given series converges by comparison to the geometric series $\sum_{n=1}^{\infty} \frac{1}{9^n}$.

6. Let A be the matrix

$$A = \begin{bmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{bmatrix},$$

where all arguments of $\cos$ are in radians. Find, with proof, $\det A$.

Solution: We claim $\det A = 0$. Recall the sum to product identities:

$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

Thus,

$$\cos 1 + \cos 3 = 2\cos 2\cos 1$$
$$\cos 4 + \cos 6 = 2\cos 5\cos 1$$
$$\cos 7 + \cos 9 = 2\cos 8\cos 1.$$

Since the sum of the first and third column is a scalar multiple of the second column, it follows that $\det A = 0$.

7. Evaluate the integral

$$\int_1^\infty \frac{\lfloor x \rfloor}{x^3} dx,$$

where $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. (Hint: You can use, without proof, the fact that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$).

Solution: Denoting the above integral by I, observe that

$$I = \sum_{k=1}^{\infty} \int_k^{k+1} \frac{k}{x^3} dx = \sum_{k=1}^{\infty} \int_0^1 \frac{k}{(k+y)^3} dy,$$

Using the FTC, we find

$$\begin{aligned} I &= -\frac{1}{2}\sum_{k=1}^{\infty} \frac{k}{(k+y)^2}\bigg|_{y=0}^{1} \\ &= -\frac{1}{2}\sum_{k=1}^{\infty}\left[\frac{k}{(k+1)^2} - \frac{1}{k}\right] \\ &= -\frac{1}{2}\sum_{k=1}^{\infty}\left[\frac{k+1}{(k+1)^2} - \frac{1}{k} - \frac{1}{(k+1)^2}\right] \\ &= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}. \end{aligned}$$

The first series above telescopes, while the other can be evaluated with the help of the given hint. We thus find

$$I = \frac{1}{2} + \frac{1}{2}\left(\frac{\pi^2}{6} - 1\right) = \frac{\pi^2}{12}.$$

8. For each positive integer $n = 1, 2, 3, \ldots$, the n^{th} Catalan number is defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Prove that C_n is a positive integer for each n.

Solution: By definition, we now that $(n+1)C_n = \left(\begin{array}{c} 2n \\ n \end{array} \right)$. Similarly, we have

$$C_n = \frac{1}{n+1} \left(\begin{array}{c} 2n \\ n \end{array} \right) = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n} \frac{(2n)!}{(n+1)!(n-1)!} = \frac{1}{n} \left(\begin{array}{c} 2n \\ n-1 \end{array} \right)$$

so that, additionally, we have $nC_n = \left(\begin{array}{c} 2n \\ n-1 \end{array} \right)$. Subtracting these two results gives

$$C_n = \left(\begin{array}{c} 2n \\ n \end{array} \right) - \left(\begin{array}{c} 2n \\ n-1 \end{array} \right).$$

This expresses C_n as the difference of two binomial coefficients, and hence C_n is always an integer. Since we clearly have

$$\left(\begin{array}{c} 2n \\ n-1 \end{array} \right) < \left(\begin{array}{c} 2n \\ n \end{array} \right)$$

for all n, it follows that C_n is additionally non-negative.

9. Given $n \in \mathbb{N}$, let $f(n)$ be the number of 1s in the binary expansion of n. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$

hint: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

Solution: Let S denote the sum of this series (absolutely convergent since $f(n) < 1 + \log_2(n)$). One can see that $f(2n) = f(n)$ and $f(2n+1) = f(n) + 1$. Using this, we split the series into even and odd parts

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{f(2n)}{(2n)(2n+1)} + \sum_{n=0}^{\infty} \frac{f(2n+1)}{(2n+1)(2n+2)} \\
&= \sum_{n=1}^{\infty} \frac{f(n)}{(2n)(2n+1)} + \sum_{n=0}^{\infty} \frac{f(n)+1}{(2n+1)(2n+2)} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} + \sum_{n=1}^{\infty} \left(\frac{f(n)}{(2n)(2n+1)} + \frac{f(n)}{(2n+1)(2n+2)} \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{n=1}^{\infty} \frac{f(n)}{2n(n+1)} \\
&= \ln 2 + \frac{S}{2}.
\end{aligned}$$

Since $S = \ln 2 + \frac{S}{2}$, we get $S = \ln 4$.

10. Let $A = \{4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, \ldots\}$ be the set of all distinct integers of the form n^k, where n and k are both integers with $n \geq 2$ and $k \geq 2$. (Note that integers such as $16 = 2^4 = 2^2$ that have multiple representations as powers are counted only once in the set A.) Evaluate, with proof, the infinite series

$$\sum_{a \in A} \frac{1}{a-1}.$$

Solution: Let B denote the set of all integers ≥ 2 that are not in A (the requirement that elements of B are not in A handles the possible "double counting" that is mentioned above). Then each element of A can be uniquely written as $a = b^k$ where $k \geq 2$ and $b \in B$. Thus,

$$\begin{aligned}\sum_{a \in A} \frac{1}{a-1} &= \sum_{b \in B} \sum_{k=2}^{\infty} \frac{1}{b^k - 1} \\ &= \sum_{b \in B} \sum_{k=2}^{\infty} \sum_{h=1}^{\infty} \frac{1}{b^{kh}},\end{aligned}$$

where we have expressed $\frac{1}{b^k - 1}$ as a geometric series. Now, switching the order of integration (this is allowed since all terms are positive!) we find

$$\begin{aligned}\sum_{a \in A} \frac{1}{a-1} &= \sum_{b \in B} \sum_{h=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{b^{kh}} \\ &= \sum_{b \in B} \sum_{h=1}^{\infty} \left(\frac{1}{b^h - 1} - \frac{1}{b^h} \right).\end{aligned}$$

Now, by definition of the set B (the set of all positive integers ≥ 2 that are not powers), the numbers b^h with h ranging over all positive integers are exactly the set of all integers $n \geq 2$ where each n occurs exactly once as $n = b^h$. Thus, the above sum is the same as

$$\sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right),$$

which is telescoping series that sums to 1. Thus,

$$\sum_{a \in A} \frac{1}{a-1} = 1.$$