2021 Kansas MAA Undergraduate Mathematics Competition and Solutions

1. Show that
$$\int_0^1 \frac{\ln(x)}{x-1}dx = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

   Solution: Integrate by parts, using $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

2. Let $a \in (0,1)$ be fixed. Determine all non-negative continuous functions f on $[0,1]$ (or prove that one does not exist) which satisfy the following three conditions:
$$\int_0^1 f(x)dx = 1, \quad \int_0^1 xf(x)dx = a, \quad \text{and} \quad \int_0^1 x^2 f(x)dx = a^2.$$

   Solution: No such function exists since then
$$\int_0^1 (a-x)^2 f(x)dx = a^2 \int_0^1 f(x)dx - 2a \int_0^1 xf(x)dx + \int_0^1 x^2 f(x)dx = 0.$$

   Since $(a-x)^2$ is non-negative, it follows that we must have $f(x) = 0$. But then this function does not satisfy the given conditions since $a > 0$. Thus, no function exists.

3. Suppose that $0 < \alpha < \beta < \gamma$. Show that the equation
$$\alpha^x + \beta^x = \gamma^x$$

   has at most one real solution.

   Solution

   Observe that $\alpha^x + \beta^x = \gamma^x \iff \left(\frac{\alpha}{\gamma}\right)^x + \left(\frac{\beta}{\gamma}\right)^x - 1 = 0$. Let $f(x) = \left(\frac{\alpha}{\gamma}\right)^x + \left(\frac{\beta}{\gamma}\right)^x - 1$ and note that this is a smooth function. Then,
$$f'(x) = \left(\frac{\alpha}{\gamma}\right)^x \log\left(\frac{\alpha}{\gamma}\right) + \left(\frac{\beta}{\gamma}\right)^x \log\left(\frac{\alpha}{\gamma}\right) < 0$$

   since $\log\left(\frac{\alpha}{\gamma}\right), \log\left(\frac{\alpha}{\gamma}\right) < 0$. Therefore, $f(x)$ is strictly increasing on $\mathbb{R}$ and hence has at most one real zero.

4. Two players A and B play a game with 2021 other persons. All 2021 people are arranged in a circle in such a way that A and B are not in initially adjacent positions. A and B play alternately, with A going first, and a play consists of choosing one of their two immediate neighbors, who is then removed from the circle. The player (A or B) who removes the other player wins the game. Describe, with proof, a winning strategy for one of the players.

Solution: A clearly can always win since one of the arcs between A and B will have an even number of people while the other will have an odd number (since 2021 is odd). At each play, player A either eliminates B (if they are neighbors), or else A eliminates the neighbor along the "odd" arc. This leaves an odd number of people on both arcs between A and B. When B chooses then, they can not choose A and hence will leave an odd and an even arc for A to choose from. The total number of persons is reduced by one after each play, so eventually B will leave zero people on one of the arcs, and A will then win by removing B.

5. Let I be the $n \times n$ identity matrix. Prove that $AB - BA \neq I$ for any $n \times n$ matrices A and B.

Solution: Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$. Then the sum of the diagonal (i.e., the trace) of $AB - BA$ is

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}b_{ji} - \sum_{j=1}^{n} b_{ij}a_{ji}\right) = 0.$$

Therefore, $AB - BA \neq I$.

6. For $n = 1, 2, 3, \dots$, let

$$I_n = \int_0^1 \frac{t^{n-1}}{(t+1)^n}\, dt.$$

(i) Show that $I_{n+1} \leqslant \frac{1}{2} I_n$.

(ii) Show also that $I_{n+1} = -\frac{1}{n\,2^n} + I_n$.

(iii) Deduce that $I_n \leqslant \frac{1}{n\,2^{n-1}}$.

(iv) Prove that $\log 2 = \sum_{r=1}^{n} \frac{1}{r\,2^r} + I_{n+1}$ and hence show that $\frac{2}{3} < \log 2 \leqslant \frac{17}{24}$.

Solution

(i) Since $0 \leqslant t \leqslant 1$ implies $\frac{t}{t+1} \leqslant \frac{1}{2}$, we have

$$I_{n+1} = \int_0^1 \frac{t^n}{(t+1)^{n+1}}\, dt = \int_0^1 \frac{t}{t+1} \cdot \frac{t^{n-1}}{(t+1)^n}\, dt \leqslant \int_0^1 \frac{1}{2} \cdot \frac{t^{n-1}}{(t+1)^n}\, dt = \frac{1}{2} I_n.$$

(ii) Integrating by parts, we have

$$I_{n+1} = \int_0^1 \frac{t^n}{(t+1)^{n+1}}\, dt = \left[-\frac{1}{n}\frac{t^n}{(t+1)^n}\right]_{t=0}^{1} + \int_0^1 \frac{t^{n-1}}{(t+1)^n}\, dt = -\frac{1}{n\,2^n} + I_n.$$

(iii) Combining parts (i) and (ii), we have

$$-\frac{1}{n\,2^n} + I_n \leqslant \frac{1}{2} I_n \;\Rightarrow\; I_n \leqslant \frac{1}{n\,2^{n-1}}.$$

(iv) From (ii), we observe that

$$\sum_{r=1}^{n} \frac{1}{r\,2^r} = \sum_{r=1}^{n} (I_r - I_{r+1}) = (I_1 - I_2) + (I_2 - I_3) + \ldots + (I_{n-1} - I_n) + (I_n - I_{n+1})$$

i.e. $\sum_{r=1}^{n} \frac{1}{r\,2^r} = I_1 - I_{n+1}$. Moreover, $I_1 = \int_0^1 \frac{1}{t+1}\,dt = [\log(t+1)]_{t=0}^1 = \log 2$.

Therefore, $\log 2 = \sum_{r=1}^{n} \frac{1}{r\,2^r} + I_{n+1}$ for all $n = 1, 2, 3, \ldots$.

In particular, for $n = 2$ we have

$$\log 2 = \sum_{r=1}^{2} \frac{1}{r\,2^r} + I_3 \overset{\text{(iii)}}{\leqslant} \frac{1}{2} + \frac{1}{8} + \frac{1}{12} = \frac{17}{24}.$$

Moreover,

$$\log 2 > \sum_{r=1}^{3} \frac{1}{r\,2^r} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} = \frac{16}{24} = \frac{2}{3}.$$

7. If a, b, and c are positive real numbers with $abc = 1$, prove that

$$a^{b+c} b^{c+a} c^{a+b} \leq 1.$$

Solution: Without loss of generality, we may assume $a \leq b \leq c$ so that, by assumption $a \leq 1$ and $c \geq 1$. Using $b = 1/(ac)$ we can rewrite

$$a^{b+c} b^{c+a} c^{a+b} = \frac{a^{b-a}}{c^{c-b}},$$

which is clearly at most one since $b - a, c - b \geq 0$.

8. The triangle OAB is isosceles, with $OA = OB$ and angle $B\hat{O}A = 2\alpha$ where $0 < \alpha < \pi/2$. The semicircle C_0 is centered at the midpoint of the base AB of the triangle, and the sides OA and OB are both tangent to the semicircle. The curves C_1, C_2, C_3, ... are circles such that C_n is tangent to C_{n-1} and also to the sides OA and OB of the triangle. Let r_n be the radius of C_n. Let S be the total area of the semicircle C_0 and the circles $C_1, C_2, C_3, \ldots$. Show that

$$S = \frac{1 + \sin^2 \alpha}{4 \sin \alpha}\, \pi r_0^2.$$

Hint: First show that

$$\frac{r_{n+1}}{r_n} = \frac{1-\sin\alpha}{1+\sin\alpha}.$$

Solution

For each $n = 0, 1, 2, \ldots$ let K_n denote the center of C_n and let L_n and R_n denote the points of contact of C_n with OA and OB, respectively. The key observation is that, since C_n is tangent to OA and OB, the triangles OL_nK_n and OK_nR_n are right-angled at L_n and R_n respectively. Moreover, since $OA = OB$, the median OK_0 is the bisector of $B\hat{O}A$ and the perpendicular bisector of AB. Therefore, OK_0 goes through the centers K_n for all $n = 1, 2, 3, \ldots$. In turn, $L_n\hat{O}K_n = K_n\hat{O}R_n = \alpha$. Then, working in the triangle OL_nK_n we have

$$r_n = (OK_n)\sin\alpha, \quad n = 1, 2, 3, \ldots .$$

Hence, since $(OK_n) = (OK_{n+1}) + r_n + r_{n+1}$ and $\alpha \in (0, \pi/2)$,

$$\frac{r_n}{\sin\alpha} = \frac{r_{n+1}}{\sin\alpha} + r_n + r_{n+1} \Leftrightarrow (1-\sin\alpha)r_n = (1+\sin\alpha)r_{n+1} \Leftrightarrow \frac{r_{n+1}}{r_n} = \frac{1-\sin\alpha}{1+\sin\alpha}.$$

Using this relation recursively, we obtain

$$r_n = \left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^n r_0, \quad n = 0, 1, 2, \ldots .$$

Therefore, we have

$$S = \frac{1}{2}\pi r_0^2 + \pi\sum_{n=1}^{\infty} r_n^2 = \frac{1}{2}\pi r_0^2 + \pi r_0^2 \sum_{n=1}^{\infty}\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^{2n} = \frac{1}{2}\pi r_0^2 + \pi r_0^2 \cdot \frac{\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2}{1-\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2}$$

which simplifies to the desired result.

9. Let

$$A = \{(a,b) \mid a, b \in \mathbb{Z}, 3a+7b \text{ is an integer multiple of } 11\}$$
$$B = \{(a,b) \mid a, b \in \mathbb{Z}, a-5b \text{ is an integer multiple of } 11\}.$$

Please prove or disprove: $A = B$.

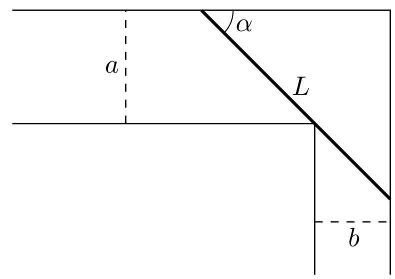
Solution

They are indeed equal, we see the divisibility criteria are logically equivalent:

$$11|(3a+7b) \iff 11|4(3a+7b) \iff 11|(4(3a+7b) - 11(a+3b)) \iff 11|(a-5b).$$

10. An attempt is made to move a rod of length L from a horizontal corridor of width a into a horizontal corridor of width b, where $a \neq b$. The corridors meet at right angles, and the rod remains horizontal. Show that if the attempt is to be successful then

$$L \leqslant \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha} \quad \text{where } \tan^3\alpha = \frac{a}{b}.$$



Solution

The attempt will fail if the rod is long enough so that at some instance its two ends are touching the outer walls of the corridors while an interior point is touching the interior corner (right angle). Then, denoting by α the (acute) angle formed by the rod and the first corridor, we have

$$\sin\alpha = \frac{a}{L_1} \;\Rightarrow\; L_1 = \frac{a}{\sin\alpha}, \quad \cos\alpha = \frac{b}{L_2} \;\Rightarrow\; L_2 = \frac{b}{\cos\alpha}$$

thus $L = L_1 + L_2 = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$. Note that this is a family of lengths which depends on α and for which the attempt fails. Thus, our condition on L such that the attempt succeeds should be that $L \leqslant L_{\text{min}}$, where L_{min} is the minimum length for which the attempt fails. Hence, we must have

$$L \leqslant \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$$

with α such that the right-hand side is minimized. This is found by differentiating with respect to α and setting to zero:

$$\frac{d}{d\alpha}\left(\frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}\right) = 0 \;\Leftrightarrow\; -\frac{a\cos\alpha}{\sin^2\alpha} + \frac{b\sin\alpha}{\cos^2\alpha} = 0 \;\Leftrightarrow\; \tan^3\alpha = \frac{a}{b}.$$

<u>Alternates</u>

11. Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{y}{x} - \frac{1}{y}.$$

Solution
Multiplying both sides by y, we have

$$y\frac{dy}{dx} = \frac{y^2}{x} - 1 \;\Rightarrow\; \frac{1}{2}\frac{d}{dx}\left(y^2\right) = \frac{y^2}{x} - 1.$$

This motivates us to set $u = y^2$, which turns our equation into

$$\frac{1}{2}\frac{du}{dx} = \frac{u}{x} - 1 \;\Rightarrow\; \frac{du}{dx} - \frac{2}{x}u = -2.$$

This is a first-order linear ODE which can be solved via the integrating factor method. In particular, we have

$$\frac{d}{dx}\left(x^{-2}u\right) = -2x^{-2} \;\Rightarrow\; x^{-2}u = -2\int x^{-2}dx = 2x^{-1} + C, \quad C = \text{const.}$$

Therefore, $u = y^2 = 2x + Cx^2$.

12. Evaluate the limit

$$L := \lim_{x\to\infty}(x+3)\int_x^{x^2} \frac{dy}{y(y+2021)^{5/2}}dy.$$

Solution: This is relatively easy, but not an easy limit to guess: Since the quantity $y(y+2021)^{5/2}$ is strictly increasing for $y > 0$ we have that

$$\int_x^{x^2} \frac{dy}{y(y+2021)^{5/2}} \leq \int_x^{x^2} \frac{dy}{x(x+2021)^{5/2}} = \frac{x^2 - x}{x(x+2021)^{5/2}}.$$

Thus, for all $x > 0$ we have

$$0 < (x+3)\int_x^{x^2} \frac{dy}{y(y+2021)^{5/2}}dy \leqslant (x+3)\frac{x^2 - x}{x(x+2021)^{5/2}}.$$

Since the right-hand side tends to 0 as $x \to \infty$, we conclude that the limit is 0 by the squeeze theorem.