2022 Kansas MAA Undergraduate Mathematics Competition

Instructions

- This is a **team competition**. You are permitted to work with the members of your team on the following 10 problems. You are not permitted to ask for or receive any assistance from anyone other than your team mates.
- Each team should submit at most one solution to each problem. Each solution should be completed on a separate sheet of paper. Also, each solution page should have the problem number **AND** the team number written on it (your team number will be communicated to you by your coach). **Please do not write your school name, the names of the team mates, or any other identifying information on the solutions.**
- The competition begins at 8 a.m. and you have until 11 a.m. to complete the problems. Once finished, each team should place the problem sheet and each solution page (with the team number and problem number written on it) into the exam envelope. Any scrap or extra paper should be returned to the graders.
- Calculators are permitted. However, you must show all of your work for full credit.
- No cell phones, other electronic devices (apart from calculators), books, notes, or any other outside help are permitted.

Problems

1. A *derangement* is a permutation with no fixed points. For example, given the 4 element set $\{A, B, C, D\}$, (B, C, A, D) is *not* a derangement since D is fixed. Similarly, (A, C, D, B) is *not* a derangement since A is fixed. However, (B, A, D, C) *is* a derangement. Given a set of n elements, let $d(n)$ denote the number of derangements of that set. Please prove, for $n > 2$, the following recursive formula for $d(n)$:

$$d(n) = (n-1) \cdot (d(n-2) + d(n-1)).$$

Solution

Consider an n-element set $\{1, 2, \ldots, n\}$. Clearly, we have $n-1$ choices for where to send 1 (namely, an element of $\{2, 3, \ldots, n\}$). For simplicity, let's say 1 is sent to 2. This is without loss of generality since we could always reorder the elements of the range so that 1 get sent to 2 under that ordering. Now either 2 gets sent to 1, or 2 gets sent to a number in the range $3, \ldots, n$. If 2 gets sent to 1, we know that $\{3, \ldots, n\}$ must map to $\{3, \ldots, n\}$ with no fixed points, of which there are

$$d(n-2)$$

derangements. On the other hand, suppose 2 gets sent to a number in the range $3, \ldots, n$. In this case, the problem reduces to counting fixed point free maps from $\{2, \ldots, n\}$ to $\{1, \ldots, n\} \setminus \{2\}$ where 2 *does not* map to 1. Evidently, this is logically equivalent to counting derangements of an $n-1$ element set,

$$d(n-1).$$

Recalling there were $n-1$ choices for where to send 1, we infer that

$$d(n) = (n-1) \cdot (d(n-2) + d(n-1)).$$

2. A rectangular strip is divided into n bands of equal length, each of which is colored with one of m colors. Two strips are considered identical if one is a left-to-right mirror reflection of the other. Please determine how many distinct colorings there are of said strips (your final answer should be a function in terms of m, n, possibly defined piecewise depending on whether n is even or odd).

1	**2**	**.............**	**n**

Solution

Either a coloring is invariant (unchanged) under reflection or it isn't.

First we count the tillings that are unchanged under reflection. If n is even, a coloring that is unchanged under reflection is completely determined by how the first $\frac{n}{2}$ tiles are colored, so there are

$$m^{\frac{n}{2}}$$

colorings invariant under reflection (when n is even).

If n is odd, the first $(n-1)/2$ tiles determine the last $(n-1)/2$ tiles, with the middle tile having full choice of the m colors. Thus, there are

$$m^{\frac{n-1}{2}} \cdot m = m^{(n+1)/2}$$

colorings invariant under reflection when n is odd.

There are m^n total colorings. If n is even, there are

$$m^n - m^{\frac{n}{2}}$$

colorings that are *not invariant* under reflection. We divide this by 2 to get the distinct colorings, and then add back in those that are invariant under reflection,

$$\frac{1}{2}(m^n - m^{\frac{n}{2}}) + m^{\frac{n}{2}} = \frac{1}{2}\left(m^n + m^{\frac{n}{2}}\right)$$

A similar process applies if n is odd,

$$\frac{1}{2}(m^n - m^{(n+1)/2}) + m^{(n+1)/2} = \frac{1}{2}\left(m^n + m^{\frac{n+1}{2}}\right).$$

Therefore, we get the piecewise function

$$C(m,n) = \begin{cases} \frac{1}{2}\left(m^n + m^{\frac{n}{2}}\right), n \textit{ is even.} \\ \frac{1}{2}\left(m^n + m^{\frac{n+1}{2}}\right), n \textit{ is odd.} \end{cases}$$

3. Do there exist integers a, b, c, d so that the last 4 digits of

$$(a+b)(b+c)(c+d)(d+a)$$

are equal to 2022? Prove the existence of such 4 numbers, or prove that such 4 numbers cannot exist.

Solution

No such integers can exist. Indeed, arguing by contradiction, suppose there were such 4 integers a, b, c, d. By divisibility tests, $(a+b)(b+c)(c+d)(d+a)$ would be even, but not divisible by 4 (since 22 is not divisible by 4).

But this means exactly one of $a+b$, $b+c$, $c+d$, $d+a$ must be even, with the other 3 being odd.

But then the sum of $a+b$, $b+c$, $c+d$, $d+a$ must be odd, an evident contradiction since

$$a+b+b+c+c+d+d+a = 2(a+b+c+d),$$

is even.

4. Let $n \in \mathbb{N}$. Let $d_1, d_2, \ldots, d_k$ denote the list of positive divisors of n. For each divisor d_i, let m_i denote the number of positive divisors of d_i. Please prove that

$$(m_1 + m_2 + \ldots + m_k)^2 = m_1^3 + m_2^3 + \ldots + m_k^3.$$

Solution

Suppose the prime factorization of n has 3 distinct primes, $n = p_1^{\ell_1} \cdot p_2^{\ell_2} \cdot p_3^{\ell_3}$.

Each triple (a_1, a_2, a_3) with $0 \le a_i \le \ell_i$ corresponds to precisely one divisor of n, namely the divisor

$$p_1^{a_1} p_2^{a_2} p_3^{a_3},$$

and the total number of positive divisors of $p_1^{a_1} p_2^{a_2} p_3^{a_3}$ is given by

$$(a_1+1)(a_2+1)(a_3+1).$$

Thus, if we sum over all such $(a_1+1)(a_2+1)(a_3+1)$, we get, using standard summation formula the sum of the first n natural numbers,

$$\begin{aligned}
m_1 + m_2 + \cdots m_k &= \sum_{i=1}^{a_1+1} \sum_{j=1}^{a_2+1} \sum_{k=1}^{a_3+1} i \cdot j \cdot k \\
&= \left(\sum_{i=1}^{a_1+1} i \right) \cdot \left(\sum_{j=1}^{a_2+1} j \right) \cdot \left(\sum_{k=1}^{a_3+1} k \right) \\
&= \frac{(a_1+1)(a_1+2)}{2} \cdot \frac{(a_2+1)(a_2+2)}{2} \cdot \frac{(a_3+1)(a_3+2)}{2}.
\end{aligned}$$

Therefore,

$$(m_1 + m_2 + \cdots m_k)^2 = \frac{(a_1+1)^2(a_1+2)^2}{4} \cdot \frac{(a_2+1)^2(a_2+2)^2}{4} \cdot \frac{(a_3+1)^2(a_3+2)^2}{4}.$$

On the other hand, using the standard summation formula for the sum of cubes,

$$\begin{aligned}
m_1^3 + m_2^3 + \cdots m_k^3 &= \sum_{i=1}^{a_1+1} \sum_{j=1}^{a_2+1} \sum_{k=1}^{a_3+1} (i \cdot j \cdot k)^3 \\
&= \left(\sum_{i=1}^{a_1+1} i^3 \right) \cdot \left(\sum_{j=1}^{a_2+1} j^3 \right) \cdot \left(\sum_{k=1}^{a_3+1} k^3 \right) \\
&= \frac{(a_1+1)^2(a_1+2)^2}{4} \cdot \frac{(a_2+1)^2(a_2+2)^2}{4} \cdot \frac{(a_3+1)^2(a_3+2)^2}{4}.
\end{aligned}$$

Thus we see the desired equality. The general case is easily seen to follow from this, where instead of three sums we get one for each prime appearing in the prime factorization of n.

5. Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2022} x}{\sin^{2022} x + \cos^{2022} x}\, dx = \frac{\pi}{4}.$$

Solution

Let $x = \frac{\pi}{2} - u$, or $u = \frac{\pi}{2} - x$. Then, since

$$\begin{aligned}
\sin x &= \sin\left(\frac{\pi}{2} - u\right) = \cos u, \\
\cos x &= \cos\left(\frac{\pi}{2} - u\right) = \sin u, \\
dx &= -du,
\end{aligned}$$

Subsituting, and changing the bounds of integration,

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{\sin^{2022} x}{\sin^{2022} x + \cos^{2022} x}\, dx \\
&= -\int_{\pi/2}^{0} \frac{\cos^{2022} u}{\cos^{2022} u + \sin^{2022} u}\, du \\
&= \int_0^{\pi/2} \frac{\cos^{2022} u}{\sin^{2022} u + \cos^{2022} u}\, du.
\end{aligned}$$

Therefore,

$$\begin{aligned}
2I &= \int_0^{\frac{\pi}{2}} \frac{\sin^{2022} x}{\sin^{2022} x + \cos^{2022} x}\, dx + \int_0^{\pi/2} \frac{\cos^{2022} u}{\sin^{2022} u + \cos^{2022} u}\, du \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^{2022} x}{\sin^{2022} x + \cos^{2022} x}\, dx + \int_0^{\pi/2} \frac{\cos^{2022} x}{\sin^{2022} x + \cos^{2022} x}\, dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^{2022} x + \cos^{2022} x}{\sin^{2022} x + \cos^{2022} x}\, dx \\
&= \int_0^{\frac{\pi}{2}} 1\, dx = \frac{\pi}{2}.
\end{aligned}$$

Therefore, the original integral I equals $\frac{\pi}{4}$.

6. Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{y}{x} - \frac{1}{y}.$$

Solution

Multiplying both sides by y, we have

$$y\frac{dy}{dx} = \frac{y^2}{x} - 1 \;\Rightarrow\; \frac{1}{2}\frac{d}{dx}\left(y^2\right) = \frac{y^2}{x} - 1.$$

This motivates us to set $u = y^2$, which turns our equation into

$$\frac{1}{2}\frac{du}{dx} = \frac{u}{x} - 1 \;\Rightarrow\; \frac{du}{dx} - \frac{2}{x}u = -2.$$

This is a first-order linear ODE which can be solved via the integrating factor method. In particular, we have

$$\frac{d}{dx}\left(x^{-2}u\right) = -2x^{-2} \;\Rightarrow\; x^{-2}u = -2\int x^{-2}dx = 2x^{-1} + C, \quad C = \text{const}.$$

Therefore, $u = y^2 = 2x + Cx^2$.

7. The nth degree polynomial $p(x)$ is called *reflexive* if all of the following hold:

- $p(x)$ is of the form $x^n - a_1x^{n-1} + a_2x^{n-2} - \ldots + (-1)^n a_n$ where $n \geqslant 1$;
- $a_1, a_2, \ldots, a_n$ are real;
- the n (not necessarily distinct) roots of $p(x)$ are $a_1, a_2, \ldots, a_n$.

(i) Find all reflexive polynomials of degree less than or equal to 3.

(ii) For any reflexive polynomial with $n = 4$, show that

$$2a_2 = -a_2^2 - a_3^2 - a_4^2.$$

Solution

(i) If $n = 1$, then $\boxed{p_1(x) = x - a_1}$ with $a_1 \in \mathbb{R}$. Note that $p_1(a_1) = 0$ as required.

If $n = 2$, then $p_2(x) = x^2 - a_1x + a_2$ with $a_1, a_2 \in \mathbb{R}$. We need $p_2(a_1) = p_2(a_2) = 0$, which implies the conditions

$$a_1^2 - a_1 \cdot a_1 + a_2 = 0, \quad a_2^2 - a_1 \cdot a_2 + a_2 = 0 \;\Rightarrow\; a_2 = 0.$$

Thus, for $n = 2$ the most general reflexive polynomial is $\boxed{p_2(x) = x^2 - a_1x} = x\,(x - a_1)$ with $a_1 \in \mathbb{R}$.

If $n = 3$, then $p_3(x) = x^3 - a_1x^2 + a_2x - a_3$ and, since the roots are a_1, a_2, a_3, we must have

$$x^3 - a_1x^2 + a_2x - a_3 = (x - a_1)(x - a_2)(x - a_3).$$

As the right-hand side is equal to

$$x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3)x - a_1a_2a_3,$$

we must have (Vieta's formulae)

$$a_1a_2a_3 = a_3, \quad a_1a_2 + a_1a_3 + a_2a_3 = a_2, \quad a_1 + a_2 + a_3 = a_1.$$

Thus, we see that if $a_3 = 0$ then $a_2 = 0$. In that case,

$$\boxed{p_3(x) = x^3 - a_1x^2} = x^2(x - a_1)$$

which is reflexive for all $a_1 \in \mathbb{R}$.

On the other hand, if $a_3 \neq 0$ then $a_1a_2 = 1$ and $a_2 + a_3 = 0$, hence

$$a_2 = \frac{1}{a_1}, \quad a_3 = -a_2 = -\frac{1}{a_1}.$$

In that case,

$$p_3(x) = x^3 - a_1x^2 + \frac{1}{a_1}x + \frac{1}{a_1}$$

and we see that, in order to have $p_3(a_1) = 0$, we must have $a_1 = -1$, which in turn implies $a_2 = -1$ and $a_3 = 1$. Thus,

$$\boxed{p_3(x) = x^3 + x^2 - x - 1.}$$

is another reflexive third-order polynomial.

(ii) If $n = 4$, then

$$p_4(x) = x^4 - a_1x^3 + a_2x^2 - a_3x + a_4.$$

From the root conditions, we have

$$x^4 - a_1x^3 + a_2x^2 - a_3x + a_4 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$$

thus, expanding the right-hand side,

$$\begin{aligned} x^4 - a_1x^3 + a_2x^2 - a_3x + a_4 &= \left[x^2 - (a_1 + a_2)x + a_1a_2\right]\left[x^2 - (a_3 + a_4)x + a_3a_4\right] \\ &= x^4 - (a_1 + a_2 + a_3 + a_4)x^3 + [a_1a_2 + a_3a_4 + (a_1 + a_2)(a_3 + a_4)]x^2 \\ &\quad - [(a_1 + a_2)a_3a_4 + (a_3 + a_4)a_1a_2]x + a_1a_2a_3a_4. \end{aligned}$$

In particular, we must have

$$a_2 + a_3 + a_4 = 0, \quad a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4 = a_2.$$

In fact, the first of these expressions leads to the simplification of the second one and so

$$a_2 + a_3 + a_4 = 0, \quad a_2a_3 + a_2a_4 + a_3a_4 = a_2.$$

Taking the square of the first equation yields the desired result in view of the second equation.

8. The distinct points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ and $R(ar^2, 2ar)$ lie on the parabola $y^2 = 4ax$, $a > 0$. The points are such that the normal to the parabola at Q and the normal to the parabola at R both pass through P.

(i) Show that $q^2 + qp + 2 = 0$.

(ii) Show that QR passes through a certain point which is independent of the choice of P, Q and R.

Solution

(i) Note that the tangent to any point of the parabola away from the origin has slope $\frac{dy}{dx}$ satisfying the equation

$$2y\frac{dy}{dx} = 4a \;\Rightarrow\; \frac{dy}{dx} = \frac{2a}{y}.$$

By differentiating, we find that the normal to the parabola at Q has slope equal to

$$m_Q = -\frac{1}{\frac{dy}{dx}} = -\frac{1}{\frac{2a}{2aq}} = -q.$$

But we also know that the normal at Q goes through P. Thus, its slope must equal

$$m_Q = \frac{y_Q - y_P}{x_Q - x_P} = \frac{2aq - 2ap}{aq^2 - ap^2} = \frac{2}{p+q}.$$

Hence, $-q = \frac{2}{p+q}$ i.e. $q^2 + pq + 2 = 0$ as desired.

(ii) The equation for QR is

$$y - y_Q = \frac{y_R - y_Q}{x_R - x_Q}(x - x_Q) \;\Rightarrow\; y - 2aq = \frac{2ar - 2aq}{ar^2 - aq^2}(x - aq^2) \;\Rightarrow\; y - 2aq = \frac{2}{r+q}(x - aq^2).$$

Rearranging gives

$$y(r+q) - 2a(rq + q^2) = 2(x - aq^2)$$

so using part (i) we find

$$y(r+q) - 2a(qr - 2 - pq) = 2\left(x + apq + 2a\right) \;\Rightarrow\; y(r+q) - 2aqr = 2x.$$

Next, note that, similarly to (i), we have

$$m_R = \frac{2}{p+r} = -r \;\Rightarrow\; r^2 + pr + 2 = 0.$$

which implies $p = -\frac{r^2+2}{r}$. But from (i) we have that $p = -\frac{q^2+2}{q}$. Hence,

$$-\frac{q^2+2}{q} = -\frac{r^2+2}{r} \;\Rightarrow\; rq^2 + 2r = qr^2 + 2q \;\Rightarrow\; qr(q-r) = 2(q-r) \;\Rightarrow\; qr = 2.$$

Then, the equation for QR becomes

$$y(r+q) - 4a = 2x$$

and so we see that the point $(-2a, 0)$ lies on QR and is independent of the choice of P, Q and R.

9. Given a sequence $w_0, w_1, w_2, \ldots$, the sequence $F_1, F_2, \ldots$ is defined by

$$F_n = w_n^2 + w_{n-1}^2 - 4w_n w_{n-1}.$$

(i) Show that $F_n - F_{n-1} = (w_n - w_{n-2})(w_n + w_{n-2} - 4w_{n-1})$ for all $n \geqslant 2$.

(ii) The sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = 1$, $u_1 = 2$ and

$$u_n = 4u_{n-1} - u_{n-2}, \quad n \geqslant 2.$$

Prove that $u_n^2 + u_{n-1}^2 = 4u_n u_{n-1} - 3$ for all $n \geqslant 1$.

(iii) The sequence $v_0, v_1, v_2, \ldots$ is defined by $v_0 = 1$ and

$$v_n^2 + v_{n-1}^2 = 4v_n v_{n-1} - 3, \quad n \geqslant 1.$$

Prove that, for each $n \geqslant 2$, either $v_n = 4v_{n-1} - v_{n-2}$ or $v_n = v_{n-2}$.

(iv) Give explicit examples of a sequence of period 2 and a sequence of period 4 that satisfy (iii).

Solution

(i) We have

$$\begin{aligned}
F_n - F_{n-1} &= w_n^2 + w_{n-1}^2 - 4w_n w_{n-1} - w_{n-1}^2 - w_{n-2}^2 + 4w_{n-1}w_{n-2} \\
&= (w_n - w_{n-2})(w_n + w_{n-2}) - 4w_{n-1}(w_n - w_{n-2}) \\
&= (w_n - w_{n-2})(w_n + w_{n-2} - 4w_{n-1}).
\end{aligned}$$

(ii) From (i) with u_n assuming the role of w_n, and using the fact that $u_n - 4u_{n-1} + u_{n-2} = 0$, we have

$$F_n - F_{n-1} = 0, \quad n \geqslant 2.$$

But note that $F_1 = u_1^2 + u_0^2 - 4u_1 u_0 = 2^2 + 1^2 - 4 \cdot 2 \cdot 1 = -3$. Thus, $F_n = -3$ for all $n \geqslant 2$, which corresponds to the desired result.

(iii) By the definition of v_n, the corresponding sequence F_n satisfies

$$F_n = -3, \quad n \geqslant 1.$$

Thus, from part (i) with v_n in the role of w_n we find

$$0 = (v_n - v_{n-2})(v_n + v_{n-2} - 4v_{n-1}), \quad n \geqslant 2,$$

which shows that either $v_n = v_{n-2}$ or $v_n = 4v_{n-1} - v_{n-2}$ as desired.

(iv) $1, 2, 1, 2, \ldots$ and $1, 2, 7, 2, 1, 2, 7, 2, \ldots$

10. Solve the integral equation

$$f(t) = t + \int_0^{\pi} f(x) \sin(x + t) dx$$

for the continuous function $f(t)$.

Solution

Since $\sin(a+b) = \sin a \cos b + \cos a \sin b$, we have

$$f(t) = t + \left(\int_0^\pi f(x)\sin(x)dx\right)\cos(t) + \left(\int_0^\pi f(x)\cos(x)dx\right)\sin(t).$$

Hence, f has the form

$$f(t) = t + A\cos(t) + B\sin(t), \quad A, B = \text{const.}$$

Plugging this into the original equation for f, we find

$$\begin{aligned} t + A\cos(t) + B\sin(t) = t + &\int_0^\pi x\sin(x+t)dx \\ &+ A\int_0^\pi \cos(x)\sin(x+t)dx + B\int_0^\pi \sin(x)\sin(x+t)dx. \end{aligned}$$

We have

$$\begin{aligned} \int_0^\pi x\sin(x+t)dx &= -x\cos(x+t)\Big|_{x=0}^{\pi} + \int_0^\pi \cos(x+t)dx \\ &= -\pi\cos(\pi+t) + \sin(\pi+t) - \sin(t) \\ &= \pi\cos(t) - 2\sin(t). \end{aligned}$$

Moreover,

$$\begin{aligned} \int_0^\pi \cos(x)\sin(x+t)dx &= \frac{1}{2}\int_0^\pi \left[\sin(x+x+t) - \sin(x-x-t)\right]dx \\ &= \frac{1}{2}\int_0^\pi \left[\sin(2x+t) + \sin(t)\right]dx \\ &= \frac{1}{2}\left[-\frac{1}{2}\cos(2\pi+t) + \frac{1}{2}\cos(t) + \pi\sin(t)\right] = \frac{\pi}{2}\sin(t). \end{aligned}$$

Finally,

$$\begin{aligned} \int_0^\pi \sin(x)\sin(x+t)dx &= \frac{1}{2}\int_0^\pi \left[\cos(x-x-t) - \cos(x+x+t)\right]dx \\ &= \frac{1}{2}\int_0^\pi \left[\cos(t) - \cos(2x+t)\right]dx = \frac{\pi}{2}\cos(t). \end{aligned}$$

Therefore, we obtain

$$t + A\cos(t) + B\sin(t) = t + \pi\cos(t) - 2\sin(t) + A\frac{\pi}{2}\sin(t) + B\frac{\pi}{2}\cos(t).$$

Matching coefficients, we find

$$A = \pi + \frac{\pi}{2}B, \quad B = -2 + \frac{\pi}{2}A \;\Rightarrow\; A = 0,\; B = -2,$$

which implies

$$f(t) = t - 2\sin t.$$