Marathon–2016

You may use a calculator. **Do not write on the test below but only on the plain paper provided.** Answers put on the form below will not be graded.

1. Sketch a graph of $y = 2 + 3(x - 4)$.
2. Write the equation of the line of slope 2 that goes through the point $P(4, 3)$.
3. Consider the parabola $y = x^2$.
   (a) Find the equation of the line between the points $A(-a, a^2)$ and $B(b, b^2)$.
   (b) In the equation you found in part (3a), find the y-intercept.
   (c) Using the proceeding answers, describe what sort of device to do arithmetic might be constructed using a parabola.
4. Consider $y = A\cos\theta + B\sin\theta$.
   (a) Let C be chosen so that $C^2 = A^2 + B^2$. Draw a right triangle the A, B, and C as lengths. Label the angle opposite B as β.
   (b) Using the notation indicated, calculate $\cos\beta$ and $\sin\beta$.
   (c) Rewrite our initial equation in terms of $\cos\beta$ and $\sin\beta$. (Hint: You might have to "factor out" a C.)
   (d) Using the formula $\cos(X - Y) = \cos(X)\cos(Y) + \sin(X)\sin(Y)$, rewrite our original equation as a single term.
   (e) Write $y = \cos\theta + \sin\theta$ as a single term.
5. Given that $\cos(X - Y) = \cos(X)\cos(Y) + \sin(X)\sin(Y)$, prove the following.
   (a) $\cos(X + Y) = \cos(X)\cos(Y) - \sin(X)\sin(Y)$.
   (b) $\cos X \cos Y = \dfrac{1}{2}\left(\cos(X - Y) + \cos(X + Y)\right)$
   (c) $\sin X \sin Y = \dfrac{1}{2}\left(\cos(X - Y) - \cos(X + Y)\right)$
   (d) $\cos^2 X = \frac{1}{2}\left(1 + \cos(2X)\right)$
6. Recall that $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.
   (a) Show that $\det\begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix} = \lambda \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
   (b) Show that $\det\begin{pmatrix} c & d \\ a & b \end{pmatrix} = -\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
   (c) Show that $\det\begin{pmatrix} \cos\theta & \sin\theta \\ \cos\phi & \sin\phi \end{pmatrix} = \sin(\phi - \theta)$.