

Number Theory

PSU Math Relays 2016

- There are 23 problems.
- For each problem, place your answer in the appropriate blank of the answer sheet provided.
- Simplify each answer as much as possible.
- You may not use a calculator on this test.

Notation and Definitions

- $\gcd(a, b)$ means the greatest common divisor of a and b .
- $\text{lcm}(a, b)$ means the least common multiple of a and b .
- Two numbers a and b are relatively prime if $\gcd(a, b) = 1$.
- $\phi(n)$ is the number of positive integers, less than or equal to n , which are relatively prime to n .
- $a \equiv b \pmod{n}$ if $a - b$ is an integer multiple of n .
- $\lfloor a \rfloor$ is the largest integer n satisfying $n \leq a$.
- $\lceil a \rceil$ is the smallest integer n satisfying $n \geq a$.

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1. Find the smallest integer greater than 1 that is relatively prime to both 210 and 308.
 2. How many numbers between 10 and 20 are relatively prime to 455?
 3. What is the exponent of 2 in the prime factorization of 168?
 4. What is the exponent of 2 in the prime factorization of 654?
 5. What is the exponent of 5 in the prime factorization of 450?
 6. What is the exponent of 11 in the prime factorization of 161051?

Let $p(n)$ denote the number of ways a positive integer can be written as the sum of positive integers less than n . For example, $3 = 1 + 1 + 1$ and $3 = 1 + 2$, so $p(3) = 2$.

7. What is $p(5)$?

8. What is $p(6)$?
9. Express the decimal $0.454545\dots$ as a fraction in lowest terms.
10. Find $\gcd(45, 105)$.
11. Find $\text{lcm}(120, 36)$.
12. Compute $7!$.
13. Compute $\frac{21!}{19!}$.
14. Compute $\binom{9}{7}$.
15. Compute $\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \cdots + \binom{10}{9} + \binom{10}{10}$.
16. Convert 85 in base 10 to base 2.
17. Convert 10101001 in base 2 to base 10.
18. How many solutions does $x^2 \equiv 4 \pmod{12}$ have, for $0 \leq x \leq 12$?
19. Find the smallest solution greater than 3 of $x^2 + 1 \equiv 10 \pmod{12}$.
20. What is the coefficient of a^2b^5 in the expansion of $(a + b)^7$?
21. What is the coefficient of $a^1b^1c^3$ in the expansion of $(a + b + c)^5$?
22. What is the sum $\sum_{n=1}^{10} n$?
23. What is the sum $\sum_{n=1}^{10} n^2 - n$?