Marathon–2018

You may use a calculator. **Do not write on the test below but only on the plain paper provided.** Answers put on the form below will not be graded.

1. Let $L(x) = \frac{4}{3-1}(x-1) + \frac{7}{3-1}(3-x)$
   (a) Calculate $L(1)$.
   (b) Calculate $L(3)$.
   (c) Find an affine (sometimes called linear) function M such that $M(\pi) = 1$ and $M(2\pi) = 7$.

2. (a) Find A and B so that $\frac{3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$.
   (b) Find A, B, and C so that $\frac{3}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$.

3. Given that $\cos^2 \frac{x}{2} = \frac{1}{2}(1+\cos x)$ and $\sin^2 \frac{x}{2} = \frac{1}{2}(1-\cos x)$, prove $\tan \frac{x}{2} = \frac{1-\cos x}{\sin x}$.

4. Suppose that f and g are functions defined on the real numbers and that α is a real number. Suppose that $f(0) = 1$, $f(\alpha) = 0$, $g(0) = 0$, and $g(\alpha) = 1$. Finally assume that $f(x-y) = f(x)f(y) + g(x)g(y)$.
   (a) Show that $f(x)^2 + g(x)^2 = 1$.
   (b) Show that $f(\alpha - x) = g(x)$.
   (c) Show that $g(\alpha - x) = f(x)$.
   (d) Show that $f(-x) = f(x)$.

5. Let $\phi(x)$ be a function defined on the positive real numbers such that: whenever $x < y$, $\phi(x) < \phi(y)$; such that $\phi(1) = 0$; such that $\phi(xy) = \phi(x) + \phi(y)$.
   (a) Show $\phi(2) > 0$.
   (b) Show that $\phi(2^2) = 2\phi(2)$.
   (c) Show that $\phi(2^n) = n\phi(2)$.
   (d) Show that $\phi(\frac{1}{x}) = -\phi(x)$.

6. Consider the series $S = \sum_{k=1}^{n} \left(\frac{4}{k} - \frac{4}{k+1} \right)$.
   (a) Write out S term-by-term for $n = 3$.
   (b) Simplify S for $n = 3$.
   (c) Simplify S for a general fixed n.
   (d) Find the value of S as n approaches infinity.

7. Calculate $\sum_{n=1}^{\infty} \frac{7}{n(n+1)}$.